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Abstract

We prove that there cannot exist square-integrable nonzero solutions to the Helmholtz equation in an axisymmetric conical domain whose vertex angle is greater than π. This implies in particular the absence of eigenvalues embedded in the essential spectrum of a large class of partial differential operators which coincide with the Laplacian in the conical domain.

1 Introduction

The purpose of this note is to prove the following result.

Theorem 1. *Let $k > 0$, $\theta \in (0, \pi/2)$ and $\Omega := \{(x, y) \in \mathbb{R}^d \times \mathbb{R};\ y > -|x| \tan\theta\}$ with $d \geq 1$ (see Figure 1). If $u \in L^2(\Omega)$ satisfies the Helmholtz equation*

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega \tag{1}$$

in the distributional sense, then $u = 0$.

This theorem is optimal in the sense that it becomes false if $\theta = 0$. Indeed it is easy to construct solutions to the Helmholtz equation which are square-integrable in a half-plane (see Remark 5).

As the well-known Rellich's uniqueness theorem [8] and succeeding results (see, e.g., [9]), this theorem points out a forbidden behavior of solutions to the Helmholtz equation in an unbounded domain. In particular, it is not related to boundary conditions (no assumption is made on the behavior of u near the boundary of Ω). Most Rellich type results involve a particular Besov space related to the boundedness of the energy flux and lead to the uniqueness of the solution to scattering problems. Our theorem involves a more restrictive functional framework: the assumption $u \in L^2(\Omega)$ rather expresses the boundedness of the total energy in Ω, which leads to the absence of so-called *trapped modes* (or *bound states*) or equivalently, the absence of eigenvalues embedded in the continuous spectrum of a large class of operators which coincide with the Laplacian in Ω. For instance, if we consider the equation

$$\Delta u + k^2 n^2 u = 0 \quad \text{in } \mathbb{R}^{d+1},$$

with a variable index of refraction $n = n(x)$ (say, bounded) such that $n = 1$ in Ω, Theorem 1 together with the unique continuation principle (see, e.g., [6]) shows that this equation has no square-integrable nonzero solutions.

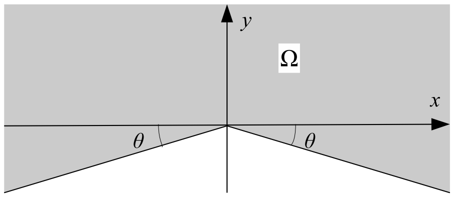

Figure 1: The conical domain Ω in the two-dimensional case ($d = 1$).

Our initial motivation concerned the possible existence of *trapped modes* in waveguides. Such solutions are known to occur for local perturbations of *closed* uniform waveguides, that is, cylindrical infinite pipes with bounded cross-section (see [7] for a review). Trapped modes also appear in the case of curved *closed* waveguides (see, e.g., [4] and more recently [3]). The situation differs singularly in the case of *open* waveguides, that is, when the transverse section becomes unbounded, as for instance optical fibers or immersed pipes. It is now understood that trapped modes do not exist in local perturbations of *open* straight waveguides [1, 5], neither in straight junctions of waveguides [2]. The example above shows that this result holds true for curved open waveguides provided that the core of the waveguide and other possible inhomogeneities are located in $\mathbb{R}^{d+1} \setminus \Omega$.

The proof of Theorem 1 is based on the two-dimensional case which is explained in sections 2 and 3. Section 4 then shows how to deal with higher dimensions as well as some possible extensions of Theorem 1. Following [1, 2, 5] (inspired by the pioneering work of Weder [10]), the main ingredients of the proof are on the one hand, a *Fourier representation* of a solution u to (1) in a half-plane and on the other hand, an *analyticity property.* The Fourier representation consists in decomposing u as a superposition of modes, which are either propagative or evanescent. Since we are only interested in square-integrable solutions, the components associated with propagative modes must vanish. As in the above mentioned papers, the fact that the components associated with evanescent modes also vanish results from the *analyticity property.* Here the key idea to obtain this property is to reuse the Fourier representation in two oblique directions.

2 Fourier representation in a half-plane

Let $\mathcal{F}$ denote the usual Fourier transform defined for a function $\varphi \in L^1(\mathbb{R})$ by

$$\mathcal{F}\varphi(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(\mathrm{X})\, \mathrm{e}^{-\mathrm{i}\mathrm{X}\xi}\, \mathrm{d}\mathrm{X} \quad \text{for } \xi \in \mathbb{R},$$

which can be classically extended to the Schwartz space of tempered distributions.

Proposition 2. *For given $\varepsilon > 0$ and $k > 0$, let u be a solution to the Helmholtz equation* (1) *in the half-plane $\Pi_\epsilon := \mathbb{R} \times (-\varepsilon, +\infty)$ such that $u \in L^2(\Pi_\epsilon)$. Then*

$$u(\mathrm{X},\mathrm{Y}) = \frac{1}{\sqrt{2\pi}} \int_{|\xi|>k} \widehat{\varphi}(\xi)\, \mathrm{e}^{\mathrm{i}\mathrm{X}\xi - \mathrm{Y}\sqrt{\xi^2-k^2}}\, \mathrm{d}\xi \quad \textit{for all } (\mathrm{X},\mathrm{Y}) \in \overline{\Pi_0}, \tag{2}$$

where $\widehat{\varphi} := \mathcal{F}u(\cdot\,,0)$ has the following properties:

$$\widehat{\varphi}(\xi) = 0 \ \textit{if } |\xi| < k \quad \textit{and} \quad \frac{\widehat{\varphi}(\xi)\, \mathrm{e}^{\varepsilon|\xi|}}{|\xi^2-k^2|^{1/4}} \in L^2(\mathbb{R}). \tag{3}$$

Remark 3. Formula (2) appears as a *modal representation* of u (analogous to the *plane wave spectrum representation* of Fourier optics) in the sense that it can be interpreted as a superposition of the modes $\exp(\mathrm{i}\mathrm{X}\xi - \mathrm{Y}\sqrt{\xi^2-k^2})$ where $\widehat{\varphi}(\xi)$ stands for the modal amplitude. These modes are propagative in the X-direction and evanescent in the Y-direction, since $|\xi| > k$ in (2). The first property in (3) expresses actually the absence of propagative modes in the Y-direction, which results from the assumption $u \in L^2(\Pi_\epsilon)$.

Remark 4. The domain Π_ϵ in which u is supposed to satisfy the Helmholtz equation is larger that the domain Π_0 where the representation (2) is written. This allows us to avoid the consequences of a possible poor regularity of u at the boundary of Π_ϵ and yields a strong decay of $\widehat{\varphi}(\xi)$ as $|\xi|$ tends to ∞, which is expressed by the exponential term in the second property of (3). Note that by Cauchy–Schwarz inequality, this property implies that $\widehat{\varphi} \in L^1(\mathbb{R})$, which shows that the integral in (2) makes sense.

Remark 5. Using the arguments of the proof below, it is readily seen that conversely to the statement of Proposition 2, if $\widehat{\varphi}$ satisfies (3), then the function u defined by (2) belongs to $L^2(\Pi_0)$ and is a solution to the Helmholtz equation (1) in Π_0. This shows that Theorem 1 is no longer true for $\theta = 0$.

Proof. As $u \in L^2(\Pi_\epsilon)$, we know that for almost every $\mathrm{Y} \in (-\varepsilon, +\infty)$, function $u(\cdot\,,\mathrm{Y})$ belongs to $L^2(\mathbb{R})$ so that we can define its Fourier transform $\widehat{u}(\xi,\mathrm{Y}) := \mathcal{F}\{u(\cdot\,,\mathrm{Y})\}(\xi)$. By the Parseval's identity,

$$\widehat{u}(\cdot\,,\mathrm{Y}) \in L^2(\mathbb{R}) \quad \text{and} \quad \|\widehat{u}(\cdot\,,\mathrm{Y})\|_{L^2(\mathbb{R})} = \|u(\cdot\,,\mathrm{Y})\|_{L^2(\mathbb{R})}.$$

As a consequence

$$\widehat{u} \in L^2(\mathbb{R} \times (-\varepsilon,+\infty)) \quad \text{and} \quad \|\widehat{u}\|_{L^2(\mathbb{R}\times(-\varepsilon,+\infty))} = \|u\|_{L^2(\Pi_\epsilon)}. \tag{4}$$

Applying the Fourier transform to the Helmholtz equation yields

$$\frac{\partial^2 \widehat{u}}{\partial \mathrm{Y}^2} + (k^2 - \xi^2)\,\widehat{u} = 0 \text{ in } \mathcal{D}'(-\varepsilon,+\infty) \quad \text{for a.e. } \xi \in \mathbb{R}.$$

Hence

$$\widehat{u}(\xi,\mathrm{Y}) = \widehat{A}(\xi)\,\mathrm{e}^{-(\mathrm{Y}+\varepsilon)\sqrt{\xi^2-k^2}} + \widehat{B}(\xi)\,\mathrm{e}^{+(\mathrm{Y}+\varepsilon)\sqrt{\xi^2-k^2}},$$

for some functions $\widehat{A}(\xi)$ and $\widehat{B}(\xi)$, where $\sqrt{\cdot}$ denotes a given determination of the complex square root such that $\sqrt{z} \in \mathbb{R}^+$ if $z \in \mathbb{R}^+$. From (4), we have $\widehat{u}(\xi,\cdot) \in L^2(-\varepsilon,+\infty)$ for almost every $\xi \in \mathbb{R}$, which implies that on the one hand, $\widehat{A}(\xi) = \widehat{B}(\xi) = 0$ if $|\xi| < k$, on the other hand, $\widehat{B}(\xi) = 0$ if $|\xi| > k$. Therefore

$$\widehat{u}(\xi,\mathrm{Y}) = \widehat{A}(\xi)\,\mathrm{e}^{-(\mathrm{Y}+\varepsilon)\sqrt{\xi^2-k^2}} \quad \text{where} \quad \widehat{A}(\xi) = 0 \text{ if } |\xi| < k. \tag{5}$$

Noticing that

$$\|\widehat{u}\|^2_{L^2(\mathbb{R}\times(-\varepsilon,+\infty))} = \int_{|\xi|>k} |\widehat{A}(\xi)|^2 \int_{-\varepsilon}^{+\infty} \mathrm{e}^{-2(\mathrm{Y}+\varepsilon)\sqrt{\xi^2-k^2}}\,\mathrm{d}\mathrm{Y}\,\mathrm{d}\xi = \int_{|\xi|>k} |\widehat{A}(\xi)|^2 \frac{\mathrm{d}\xi}{2\sqrt{\xi^2-k^2}},$$

we infer that $|\xi^2-k^2|^{-1/4}\,\widehat{A}(\xi) \in L^2(\mathbb{R})$. Setting $\widehat{\varphi}(\xi) := \widehat{A}(\xi)\,\mathrm{e}^{-\varepsilon\sqrt{\xi^2-k^2}}$ and using the inverse Fourier transform of (5), the conclusion follows. $\square$

The following corollary plays an essential role in the proof of Theorem 1.

Corollary 6. *For any half-line $\Lambda_\alpha := \{(t\cos\alpha, t\sin\alpha) \in \mathbb{R}^2;\ t > 0\}$ where $\alpha \in (0,\pi)$, a solution $u \in L^2(\Pi_\epsilon)$ to the Helmholtz equation (1) in Π_ϵ is such that $u|_{\Lambda_\alpha} \in L^1(\Lambda_\alpha)$.*

Proof. By the Fourier representation (2) of u, we have

$$\begin{aligned}
\int_0^{+\infty} |u(t\cos\alpha, t\sin\alpha)|\,\mathrm{d}t &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \left| \int_{|\xi|>k} \widehat{\varphi}(\xi)\,\mathrm{e}^{\mathrm{i}\xi\,t\cos\alpha - \sqrt{\xi^2-k^2}\,t\sin\alpha}\,\mathrm{d}\xi \right| \mathrm{d}t \\
&\leq \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \int_{|\xi|>k} |\widehat{\varphi}(\xi)|\,\mathrm{e}^{-\sqrt{\xi^2-k^2}\,t\sin\alpha}\,\mathrm{d}\xi\,\mathrm{d}t \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{|\xi|>k} \frac{|\widehat{\varphi}(\xi)|}{\sqrt{\xi^2-k^2}\,\sin\alpha}\,\mathrm{d}\xi,
\end{aligned}$$

using Fubini's theorem. We deduce from the Cauchy–Schwarz inequality that

$$\int_0^{+\infty} |u(t\cos\alpha, t\sin\alpha)|\,\mathrm{d}t \leq \frac{1}{\sqrt{2\pi}} \left\| \frac{\widehat{\varphi}(\xi)\,\mathrm{e}^{\varepsilon|\xi|}}{|\xi^2-k^2|^{1/4}} \right\|_{L^2(\mathbb{R})} \left(\int_{|\xi|>k} \frac{\mathrm{e}^{-2\varepsilon|\xi|}}{\sqrt{\xi^2-k^2}\,\sin^2\alpha}\,\mathrm{d}\xi \right)^{1/2},$$

where the right-hand side is bounded, according to (3). $\square$

3 Proof of Theorem 1 in the two-dimensional case $(d = 1)$

The proof is based on three uses of Proposition 2. In order to simplify the presentation as regards the role of ε in this proposition, we redefine the domain Ω of Theorem 1 as $\Omega := \{(x, y) \in \mathbb{R}^2;\ y + \ell > -|x| \tan\theta\}$ for some $\ell > 0$ (which amounts to a simple change of variable).

In the first use of Proposition 2, we simply notice that u is a square-integrable solution to the Helmholtz equation in the half-plane $\{y > -\ell\}$. Hence, the conclusions of the proposition hold true with $(\mathrm{X}, \mathrm{Y}) = (x, y)$ and $\varepsilon = \ell$. In particular,

$$\widehat{\varphi} := \mathcal{F}u(\cdot\,, 0) \text{ is such that } \widehat{\varphi}(\xi) = 0 \text{ if } |\xi| < k. \tag{6}$$

The key argument consists in proving that $\widehat{\varphi}(\xi)$ extends to an analytic function in a complex vicinity of the real axis. As $\widehat{\varphi}$ vanishes on the interval $(-k, +k)$, analyticity implies that it must vanish everywhere, that is, $\widehat{\varphi}(\xi) = 0$ for all ξ (recall that the zeros of an analytic function are isolated). The Fourier representation (2) then tells us that u vanishes in the half-plane $\{y > 0\}$, so also in the whole domain Ω by virtue of the unique continuation principle, which completes the proof of Theorem 1.

It remains to prove the analyticity of $\widehat{\varphi}(\xi)$. To do this, the trick is to split the definition $\widehat{\varphi} := \mathcal{F}u(\cdot\,, 0)$ in the form

$$\widehat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} u(x, 0)\, \mathrm{e}^{-\mathrm{i}x\xi}\, \mathrm{d}x + \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} u(x, 0)\, \mathrm{e}^{-\mathrm{i}x\xi}\, \mathrm{d}x \tag{7}$$

and to express $u(\cdot\,, 0)|_{\mathbb{R}^+}$ and $u(\cdot\,, 0)|_{\mathbb{R}^-}$ by using again Proposition 2 in two half-planes contained in Ω defined respectively by the equations $y > -x \tan\theta$ and $y > +x \tan\theta$. Using both changes of variables

$$\begin{pmatrix} \mathrm{X} \\ \mathrm{Y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \mp\sin\theta \\ \pm\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

which map respectively the half-planes $\{y > \mp x \tan\theta\}$ to Π_0, we obtain the following Fourier representations:

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{|\eta|>k} \widehat{\varphi}^{\pm}(\eta)\, \mathrm{e}^{\mathrm{i}\eta(x \cos\theta \mp y \sin\theta) - \sqrt{\eta^2 - k^2}\,(\pm x \sin\theta + y \cos\theta)}\, \mathrm{d}\eta \quad \text{if } y > \mp x \tan\theta,$$

where $\widehat{\varphi}^{\pm}$ both satisfy (3) with $\varepsilon = \ell \cos\theta$. Taking $y = 0$ yields $u(\cdot\,, 0)|_{\mathbb{R}^{\pm}}$. Note that Corollary 6 tells us that $u(\cdot\,, 0)|_{\mathbb{R}^{\pm}} \in L^1(\mathbb{R}^{\pm})$. This justifies equation (7) which becomes

$$\widehat{\varphi}(\xi) = \frac{1}{2\pi} \sum_{\pm} \int_{\mathbb{R}^{\pm}} \int_{|\eta|>k} \widehat{\varphi}^{\pm}(\eta)\, \mathrm{e}^{x(\mathrm{i}\eta \cos\theta \mp \sqrt{\eta^2 - k^2} \sin\theta)}\, \mathrm{d}\eta\, \mathrm{e}^{-\mathrm{i}x\xi}\, \mathrm{d}x.$$

Using the same argument as in the proof of Corollary 6, it is readily seen that the integrands of the above integrals belong respectively to $L^1(\mathbb{R}^{\pm} \times \{|\eta| > k\})$. Hence, Fubini's theorem yields

$$\widehat{\varphi}(\xi) = \frac{1}{2\pi} \sum_{\pm} \int_{|\eta|>k} \widehat{\varphi}^{\pm}(\eta) \int_{\mathbb{R}^{\pm}} \mathrm{e}^{x(\mathrm{i}(\eta \cos\theta - \xi) \mp \sqrt{\eta^2 - k^2} \sin\theta)}\, \mathrm{d}x\, \mathrm{d}\eta = \int_{|\eta|>k} F(\eta, \xi)\, \mathrm{d}\eta,$$

where

$$F(\eta, \xi) := \frac{1}{2\pi} \sum_{\pm} \frac{\widehat{\varphi}^{\pm}(\eta)}{\mp\mathrm{i}(\eta \cos\theta - \xi) + \sqrt{\eta^2 - k^2} \sin\theta}.$$

For almost every η, this function extends to an analytic function of ξ in any complex domain in which both denominators do not vanish. The complex values of ξ for which there exists a $\eta \in \mathbb{R} \setminus (-k, +k)$ such that one of the denominators vanishes is the hyperbola defined by the equation

$$\frac{(\operatorname{Re}\xi)^2}{\cos^2\theta} - \frac{(\operatorname{Im}\xi)^2}{\sin^2\theta} = k^2$$

represented in Figure 2. Hence, for almost every η, $F(\eta, \xi)$ is an analytic function of ξ in the three connected components of the complex plane delimited by this hyperbola. By the Lebesgue's dominated convergence theorem (recall that $\widehat{\varphi}^{\pm} \in L^1(\mathbb{R})$, see Remark 4), we deduce that the same holds true for $\widehat{\varphi}(\xi)$. In fact, we are interested in the analyticity of $\widehat{\varphi}(\xi)$ in the components colored in gray in Figure 2. Indeed, as we already know that $\widehat{\varphi}$ vanishes on $(-k, +k)$ (see (6)), the analyticity in these gray components implies that $\widehat{\varphi}(\xi)$ also vanishes for $\xi \in \mathbb{R} \setminus (-k, +k)$, which is the desired result.

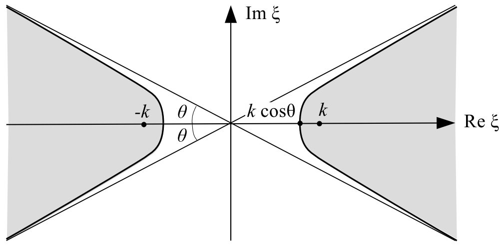

Figure 2: Domains of analyticity of $\widehat{\varphi}$

4 Higher dimensions and other extensions

The above proof of Theorem 1 is specific to the two-dimensional case, since it cannot be extended directly to higher dimensions (more precisely, Proposition 2 can readily be extended, but not the arguments of section 3). Fortunately, the case $d > 1$ is easily derived from the case $d = 1$ as follows. Suppose that $u \in L^2(\Omega)$ satisfies the Helmholtz equation (1) in Ω. We split the coordinate $x \in \mathbb{R}^d$ as $x = (x_1, x')$ where $x' := (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$ so that the Fourier transform $\mathcal{F}$ in the x variable appears as the product of the partial Fourier transforms in the x_1 and x' variables, denoted by $\mathcal{F}_1$ and $\mathcal{F}'$ respectively. Define $u'(x_1, \xi', y) := \mathcal{F}'\{u(x_1, \cdot\,, y)\}(\xi')$ and $\hat{u}(\xi, y) := \mathcal{F}\{u(\cdot\,, y)\}(\xi)$ where $\xi' \in \mathbb{R}^{d-1}$ and $\xi = (\xi_1, \xi') \in \mathbb{R}^d$ are the Fourier variables associated respectively with x' and x. By the Parseval's identity, for a.e. $\xi' \in \mathbb{R}^{d-1}$, $u'(\cdot\,, \xi', \cdot)$ is square-integrable in the cone $\Omega_{2\text{D}} := \{(x_1, y) \in \mathbb{R}^2;\ y > -|x_1| \tan\theta\}$ and satisfies

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y^2} + k^2 - |\xi'|^2\right) u'(\cdot\,, \xi', \cdot) = 0 \quad \text{in } \Omega_{2\text{D}}.$$

Hence Theorem 1 for $d = 1$ applies provided $|\xi'| < k$: in this case, we thus know that $u'(\cdot\,, \xi', \cdot)$ vanishes in $\Omega_{2\text{D}}$. As $\hat{u}(\xi_1, \xi', y) = \mathcal{F}_1\{u'(\cdot\,, \xi', y)\}(\xi_1)$, this shows that for all $y \geq 0$, $\hat{u}(\cdot\,, y)$ vanishes in the cylinder $\{(\xi_1, \xi') \in \mathbb{R}^d;\ |\xi'| < k\}$. This conclusion actually remains true for any axisymmetric cylinder of radius k whose axis contains the origin, since in the above lines, one can replace the particular direction x_1 by any direction of the x-space using a rotation. Therefore $\hat{u}(\xi, y) = 0$ for all $\xi \in \mathbb{R}^d$ and $y \geq 0$. Applying $\mathcal{F}^{-1}$ yields $u = 0$ in $\mathbb{R}^d \times \mathbb{R}^+$ and we conclude again by the unique continuation principle.

Theorem 1 also applies in an anisotropic medium described by the equation

$$\operatorname{div}(\mathbb{A}\, \mathbf{grad}\, u) + k^2\, u = 0 \quad \text{in } \Omega,$$

where $\mathbb{A}$ is a constant real symmetric positive definite $d \times d$ matrix. Indeed this equation can be transformed into our original Helmholtz equation using first the new coordinate system corresponding to a unitary matrix which diagonalizes $\mathbb{A}$, then a suitable dilation in each direction. This process transforms Ω into a new conical domain that is no longer axisymmetric in general, but that still contains an axisymmetric cone with vertex angle greater than π, which allows us to conclude.

Another application concerns the time-harmonic Maxwell's equations

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - k^2\,\mathbf{U} = 0 \quad \text{in } \Omega \subset \mathbb{R}^3.$$

In this case, we simply have to notice that, thanks to the relation $\boldsymbol{\Delta} = -\mathbf{curl}\,\mathbf{curl} + \mathbf{grad}\,\operatorname{div}$, if $\mathbf{U} \in L^2(\Omega)^3$ satisfies this equation, then each of its components $\mathbf{U}_i$ belongs to $L^2(\Omega)$ and satisfies the Helmholtz equation (1).

Theorem 1 can also be extended to some situations which involve non-homogeneous media, using a generalized Fourier transform instead of the usual one, as shown in [1, 2, 5]. Works on this subject are in progress.

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