Forward Feynman-Kac type representation for semilinear nonconservative Partial Differential Equations

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Abstract

We propose a nonlinear forward Feynman-Kac type equation, which represents the solution of a non-conservative semilinear parabolic Partial Differential Equations (PDE). We show in particular existence and uniqueness. The solution of that type of equation can be approached via a weighted particle system.

Key words and phrases: Semilinear Partial Differential Equations; Nonlinear Feynman-Kac type functional; Particle systems; Probabilistic representation of PDEs.

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1 Introduction

This paper situates in the framework of forward probabilistic representations of nonlinear PDEs of the form

$$\begin{cases} \partial_t u = \frac{1}{2}\sum_{i,j=1}^{d} \partial_{ij}^2 \left((\Phi\Phi^t)_{i,j}(t,x,u)u \right) - div\left(g(t,x,u)u \right) + \Lambda(t,x,u,\nabla u)u \,, \quad \text{for any } t\in[0,T]\,, \\ u(0,dx) = u_0(dx), \end{cases} \tag{1.1}$$

where u_0 is a Borel probability measure. $u :]0,T] \times \mathbb{R}^d \to \mathbb{R}$ will be the unknown function. For that PDE we propose the forward probabilistic representation

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(s,Y_s,u(s,Y_s))dW_s + \int_0^t g(s,Y_s,u(s,Y_s))ds \,, \quad \text{with} \quad Y_0 \sim u_0 \,, \\ u(t,\cdot) := \frac{d\nu_t}{dx} \quad \nu_t(\varphi) := \mathbb{E}\left[\varphi(Y_t) \, \exp\left\{ \int_0^t \Lambda\big(s,Y_s,u(s,Y_s),\nabla u(s,Y_s)\big)ds \right\} \right] \,, \varphi \in \mathcal{C}_b(\mathbb{R}^d,\mathbb{R}), t\in[0,T], \end{cases} \tag{1.2}$$

which is a nonlinear stochastic differential equation (NLSDE) in the spirit of McKean, see e.g. [19].

The underlying idea of our approach consists in extending, to fairly general non-conservative PDEs, the probabilistic representation of nonlinear Fokker-Planck equations which appears when $\Lambda = 0$. An interesting aspect of this strategy is that it is potentially able to represent an extended class of second order nonlinear PDEs.

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When $\Lambda = 0$, several authors have studied NLSDEs of the form (1.2). Significant contributions are due to [26], [21], [20], in the case where the non linearity with respect to u are mollified in the diffusion and drift coefficients. In [15], the authors focused on the case when the coefficients depend pointwisely on u. The authors have proved strong existence and pathwise uniqueness of (1.2), when Φ and g are smooth and Lipschitz and Φ is non-degenerate. Other authors have more particularly studied an NLSDE of the form

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(u(s, Y_s))dW_s, \ Y_0 \sim u_0, \\ u(t,x)dx \text{ is the law density of } Y_t, t > 0 \\ u(0, \cdot) = u_0 \ , \end{cases} \tag{1.3}$$

for which the particular case $d = 1$, $\Phi(u) = u^k$ for $k \geq 1$ was developed in [3]. When $\Phi : \mathbb{R} \to \mathbb{R}$ is only assumed to be bounded, measurable and monotone, existence/uniqueness results are still available in [5, 1]. A partial extension to the multidimensional case ($d \geq 2$) is exposed in [2].

The solutions of (1.1), when $\Lambda = 0$, are probability measures dynamics which often describe the macroscopic distribution law of a microscopic particle which behaves in a diffusive way. For that reason, those time evolution PDEs are conservative in the sense that their solutions $u(t, \cdot)$ verify the property $\int_{\mathbb{R}^d} u(t,x)dx$ to be constant in t (equal to 1, which is the mass of a probability measure). An interesting feature of this type of representation is that the law of the solution Y of the NLSDE can be characterized as the limiting empirical distribution of a large number of interacting particles. This is a consequence of the so called *propagation of chaos* phenomenon, already observed in the literature for the case of mollified dependence, see e.g. [16, 19, 26, 21] and [15] for the case of pointwise dependence. [8] has contributed to develop stochastic particle methods in the spirit of McKean to approach a PDE related to Burgers equation providing first the rate of convergence. Comparison with classical numerical analysis techniques was provided by [7].

In this paper we will concentrate on the novelty constituted by the introduction of Λ depending on u and ∇u. For this step Φ, g will not depend on u. In this context we will focus on semilinear PDEs of the form

$$\begin{cases} \partial_t u = L_t^* u + u\Lambda(t, x, u, \nabla u) \\ u(0, \cdot) = u_0 \ , \end{cases} \tag{1.4}$$

with L^* a partial differential operator of the type

$$(L_t^* \varphi)(x) = \frac{1}{2} \sum_{i,j=1}^{d} \partial_{ij}^2 ((\Phi\Phi^t)_{i,j}(t,x)\varphi)(x) - \sum_{i=1}^{d} \partial_i (g_i(t,x)\varphi)(x), \quad \text{for } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d). \tag{1.5}$$

An alternative approach for representing this type of PDE is given by forward-backward stochastic differential equations. Those were initially developed in [23], see also [22] for a survey and [24] for a recent monograph on the subject. However, the extension of those equations to fully nonlinear PDEs still requires complex developments and is the subject of active research, see for instance [9]. Branching diffusion processes provide another probabilistic representation of semilinear PDEs, see e.g. [11, 13, 12]. Here again, extensions to second order nonlinear PDEs still constitutes a difficult issue.

As suggested, our method potentially allows to reach a certain significant class of PDEs with second-order non-linearity, if we allow the diffusion coefficient to also depend on u. The general framework where g and Φ also depend non linearly on u while Λ depends on u and ∇u has been partially investigated in [18], where the dependence of the coefficients with respect to u is mollified and Λ does not depend on ∇u. An associated interacting particle system converging to the solution of a regularized version of the nonlinear PDE has been proposed in [17], providing encouraging numerical performances. The originality of the present paper is to consider a pointwise dependence of Λ on both u and ∇u. The pointwise dependence on ∇u constitutes the major technical difficulty.

More specifically, we propose to associate (1.4) with a forward probabilistic representation given by a couple (Y, u) solution of (1.2) where Φ and g are the functions intervening in (1.5). In this case, the second line equation of (1.2) will be called **Feynman-Kac equation** and a solution $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ will be called **Feynman-Kac type representation** of (1.4). When Λ vanishes, the functions $(u(t,\cdot), t > 0)$ are indeed the marginal law densities of the process $(Y_t, t > 0)$ and (1.4) coincides with the classical Fokker-Planck PDE. When $\Lambda \neq 0$, the proof of well-posedness of the Feynman-Kac equation is not obvious and it is one of the contributions of the paper. The strategy used relies on two steps. Under a Lipschitz condition on Λ in Theorem 3.5, we first prove that a function $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ (belonging to $L^1([0,T], W^{1,1}(\mathbb{R}^d))$) is a solution of the Feynman-Kac equation (1.2) if and only if it is a mild solution of (1.4). The latter concept is introduced in item 2. of Definition 2.1. Then, under Lipschitz type conditions on Φ and g, Theorem 3.6 establishes the existence and uniqueness of a mild solution of (1.4). As a second contribution, we propose and analyze a corresponding particle system. This relies on two approximation steps: a regularization procedure based on a kernel convolution and the law of large numbers. The convergence of the particle system is stated in Theorem 5.3.

2 Preliminaries

2.1 Notations

Let $d \in \mathbb{N}^\star$. Let us consider $\mathcal{C}^d := \mathcal{C}([0,T], \mathbb{R}^d)$ metricized by the supremum norm $\|\cdot\|_\infty$, equipped with its Borel $\sigma-$ field $\mathcal{B}(\mathcal{C}^d)$ and endowed with the topology of uniform convergence.
If (E, d_E) is a Polish space, $\mathcal{P}(E)$ denotes the Polish space (with respect to the weak convergence topology) of Borel probability measures on E naturally equipped with its Borel σ-field $\mathcal{B}(\mathcal{P}(E))$. The reader can consult Proposition 7.20 and Proposition 7.23, Section 7.4 Chapter 7 in [4] for more exhaustive information. When $d = 1$, we often omit it and we simply note $\mathcal{C} := \mathcal{C}^1$. $\mathcal{C}_b(E)$ denotes the space of bounded, continuous real-valued functions on E.

In this paper, $\mathbb{R}^d$ is equipped with the Euclidean scalar product $\cdot$ and $|x|$ stands for the induced norm for $x \in \mathbb{R}^d$. The gradient operator for functions defined on $\mathbb{R}^d$ is denoted by ∇. If a function u depends on a variable $x \in \mathbb{R}^d$ and other variables, we still denote by ∇u the gradient of u with respect to x, if there is no ambiguity. $M_{d,p}(\mathbb{R})$ denotes the space of $\mathbb{R}^{d\times p}$ real matrices equipped with the Frobenius norm (also denoted $|\cdot|$), i.e. the one induced by the scalar product $(A, B) \in M_{d,p}(\mathbb{R}^d) \times M_{d,p}(\mathbb{R}) \mapsto Tr(A^t B)$ where A^t stands for the transpose matrix of A and Tr is the trace operator. $\mathcal{S}_d$ is the set of symmetric, non-negative definite $d \times d$ real matrices and $\mathcal{S}_d^+$ the set of strictly positive definite matrices of $\mathcal{S}_d$.
$\mathcal{M}_f(\mathbb{R}^d)$ is the space of finite Borel measures on $\mathbb{R}^d$. When it is endowed with the weak convergence topology, $\mathcal{B}(\mathcal{M}_f(\mathbb{R}^d))$ stands for its Borel σ-field. It is well-known that $(\mathcal{M}_f(\mathbb{R}^d), \|\cdot\|_{TV})$ is a Banach space, where $\|\cdot\|_{TV}$ denotes the total variation norm.
$\mathcal{S}(\mathbb{R}^d)$ is the space of Schwartz fast decreasing test functions and $\mathcal{S}'(\mathbb{R}^d)$ is its dual. $\mathcal{C}_b(\mathbb{R}^d)$ is the space of bounded, continuous functions on $\mathbb{R}^d$ and $\mathcal{C}_0^\infty(\mathbb{R}^d)$ the space of smooth functions with compact support. For any positive integers $p, k \in \mathbb{N}$, $C_b^{k,p} := C_b^{k,p}([0,T] \times \mathbb{R}^d, \mathbb{R})$ denotes the set of continuously differentiable bounded functions $[0,T] \times \mathbb{R}^d \to \mathbb{R}$ with uniformly bounded derivatives with respect to the time variable t (resp. with respect to space variable x) up to order k (resp. up to order p). In particular, for $k = p = 0$, $C_b^{0,0}$ coincides with the space of bounded, continuous functions also denoted by $\mathcal{C}_b$. $\mathcal{C}_b^\infty(\mathbb{R}^d)$ is the space of

bounded and smooth functions. $\mathcal{C}_0(\mathbb{R}^d)$ denotes the space of continuous functions with compact support in $\mathbb{R}^d$. For $r \in \mathbb{N}$, $W^{r,p}(\mathbb{R}^d)$ is the Sobolev space of order r in $(L^p(\mathbb{R}^d), ||\cdot||_p)$, with $1 \leq p \leq \infty$. $W^{1,1}_{loc}(\mathbb{R}^d)$ denotes the space of functions $f : \mathbb{R}^d \to \mathbb{R}$ such that f and ∇f (existing in the weak sense) belong to $L^1_{loc}(\mathbb{R}^d)$.
For convenience we introduce the following notation.

- $V : [0,T] \times \mathcal{C}^d \times \mathcal{C} \times \mathcal{C}^d$ is defined for any functions $x \in \mathcal{C}^d$, $y \in \mathcal{C}$ and $z \in \mathcal{C}^d$, by

$$V_t(x,y,z) := \exp\left(\int_0^t \Lambda(s, x_s, y_s, z_s)ds\right) \quad \text{for any } t \in [0,T] \,. \tag{2.1}$$

The finite increments theorem gives, for all $(a,b) \in \mathbb{R}^2$, we have

$$\exp(a) - \exp(b) = (b-a)\int_0^1 \exp(\alpha a + (1-\alpha)b)d\alpha. \tag{2.2}$$

Therefore, if Λ is supposed to be bounded and Lipschitz w.r.t. to its space variables (x,y,z), uniformly w.r.t. t, we observe that (2.2) implies it follows that, for all $t \in [0,T]$, $x, x' \in \mathcal{C}^d$, $y, y' \in \mathcal{C}$, $z, z' \in \mathcal{C}^d$,

$$|V_t(x,y,z) - V_t(x',y',z')| \leq L_\Lambda e^{tM_\Lambda} \int_0^t \big(|x_s - x'_s| + |y_s - y'_s| + |z_s - z'_s|\big)ds \,, \tag{2.3}$$

M_Λ (resp. L_Λ) denoting an upper bound of $|\Lambda|$ (resp. the Lipschitz constant of Λ), see also Assumption 2.
In the whole paper, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ will denote a filtered probability space and W an $\mathbb{R}^p$-valued $(\mathcal{F}_t)$-Brownian motion.

2.2 Mild and Weak solutions

We first introduce the following assumption.

Assumption 1. *1. Φ and g are functions defined on $[0,T] \times \mathbb{R}^d$ taking values in $M_{d,p}(\mathbb{R}^d)$ and $\mathbb{R}^d$. There exist $L_\Phi, L_g > 0$ such that for any $t \in [0,T]$, $(x,x') \in \mathbb{R}^d \times \mathbb{R}^d$,*

$$\begin{aligned} |\Phi(t,x) - \Phi(t,x')| &\leq L_\Phi |x - x'| \,, \\ |g(t,x) - g(t,x')| &\leq L_g |x - x'| \,. \end{aligned}$$

2. The functions $s \in [0,T] \mapsto |\Phi(s,0)|$ and $s \in [0,T] \mapsto |g(s,0)|$ are bounded.

In the whole paper we will write $a = \Phi\Phi^t$; in particular $a : [0,T] \times \mathbb{R}^d \longrightarrow \mathcal{S}_d$. Through some definitions, we make here precise in which sense we will consider solutions of the PDE (1.4). We are interested in different concepts of solutions $u : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ of that semilinear PDE where, for $t \in [0,T]$, L_t is given by

$$(L_t\varphi)(x) = \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(t,x)\partial^2_{ij}\varphi(x) + \sum_{i=1}^d g_i(t,x)\partial_i\varphi(x), \ \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d). \tag{2.4}$$

Its "adjoint" L_t^* defined in (1.5), verifies

$$\int_{\mathbb{R}^d} L_t\varphi(x)\psi(x)dx = \int_{\mathbb{R}^d} \varphi(x)L_t^*\psi(x)dx \,, \ (\varphi,\psi) \in \mathcal{C}_0^\infty(\mathbb{R}^d), t \in [0,T] \,, \tag{2.5}$$

and the initial condition u_0 of (1.4) has to be understood in the sense that

$$\lim_{t\to 0}\int_{\mathbb{R}^d} \varphi(x)u_t(dx) = \int_{\mathbb{R}^d} \varphi(x)u_0(dx) \,, \text{ for all } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d) \,,$$

since, a priori, it can be irregular and not necessarily a function.
We observe that if $\Lambda = 0$, (1.4) is the classical Fokker-Planck equation which can be understood in the sense of distributions, i.e., for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $t \in [0,T]$,

$$\int_{\mathbb{R}^d} u(t,dx)\varphi(x) = \int_{\mathbb{R}^d} u_0(dx)\varphi(x) + \int_0^t \int_{\mathbb{R}^d} u(s,dx)(L_s\varphi)(x)ds \; . \tag{2.6}$$

Under Assumption 1 it is well-known (see Introduction and Section 2.2, Chapter 2 in [25]), that there exists a *good* family of probability transition $P(s,x_0,t,\cdot)$ (see Introduction and Section 2.2, Chapter 2 in [25]), for which the Fokker-Planck equation (understood in the sense of distributions) is verified, i.e.

$$\begin{cases} \partial_t P(s,x_0,t,\cdot) = L_t^* P(s,x_0,t,\cdot) \\ \lim_{t\downarrow s} P(s,x_0,t,\cdot) = \delta_{x_0}, \quad 0 \le s < t \le T, x_0 \in \mathbb{R}^d \; . \end{cases} \tag{2.7}$$

Given a random variable Y_0, classical theorems for SDE with Lipschitz coefficients imply strong existence and pathwise uniqueness for the SDE

$$dY_t = \Phi(t,Y_t)dW_t + g(t,Y_t)dt. \tag{2.8}$$

By the classical theory of Markov processes (see e.g. Chapter 2 in [25]), we know that the transition probability function P, satisfying (2.7), defines and characterizes uniquely the law of the process Y, provided the law u_0 of Y_0 is specified. In particular, we have the following.

1. For $t \in [0,T]$, the marginal law of Y_t is given for all $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$ by

$$\mathbb{E}[\varphi(Y_t)] \quad = \quad \int_{\mathbb{R}^d} u_0(dx_0) \int_{\mathbb{R}^d} \varphi(x)P(0,x_0,t,dx) \; . \tag{2.9}$$

2. For all $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$ and $0 \le s < t \le T$,

$$\mathbb{E}[\varphi(Y_t)|Y_s] = \int_{\mathbb{R}^d} \varphi(x)P(s,Y_s,t,dx). \tag{2.10}$$

Let $\Lambda : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be bounded, Borel measurable, we recall the notions of **weak solution** and **mild solution** associated to (1.4).

Definition 2.1. *Let $u : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be a Borel function such that for every $t \in]0,T]$, $u(t,\cdot) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$.*

1. *u will be called* **weak solution** *of* (1.4) *if for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $t \in [0,T]$,*

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x)u(t,x)dx - \int_{\mathbb{R}^d} \varphi(x)u_0(dx) \quad &= \quad \int_0^t \int_{\mathbb{R}^d} u(s,x)L_s\varphi(x)dxds \\ &+ \int_0^t \int_{\mathbb{R}^d} \varphi(x)\Lambda(s,x,u(s,x),\nabla u(s,x))u(s,x)dxds \; . \end{aligned} \tag{2.11}$$

2. *u will be called* **mild solution** *of* (1.4) *if for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $t \in [0,T]$,*

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x)u(t,x)dx \quad &= \quad \int_{\mathbb{R}^d} \varphi(x) \int_{\mathbb{R}^d} u_0(dx_0)P(0,x_0,t,dx) \\ &+ \int_{[0,t]\times\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \varphi(x)P(s,x_0,t,dx) \Big) \Lambda(s,x_0,u(s,x_0),\nabla u(s,x_0))u(s,x_0)dx_0ds \; . \end{aligned} \tag{2.12}$$

The object of the first lemma below is to show to what extent the concept of mild solution is equivalent to the weak one.

Lemma 2.2. *We assume there exists a unique weak solution v of*

$$\begin{cases} \partial_t v = L_t^* v \\ v(0,\cdot) = 0 \,, \end{cases} \tag{2.13}$$

where L_t^* is given by (1.5). *Then, u is a mild solution of* (1.4) *if and only if u is a weak solution of* (1.4)

Remark 2.3. *There exist several sets of technical assumptions (see e.g. [6, 10]) leading to the uniqueness assumed in Lemma 2.2 above. In particular, under items 1., 2. and 3. of Assumption 2 stated below (which will constitutes our framework in the sequel), Theorem 4.7 in Chapter 4 of [10] ensures (classical) existence and uniqueness of the solution of* (2.13)*, see also Lemma 6.4 below.*

Proof. We first suppose that u is a mild solution of (1.4). Taking into account that $P(s, x_0, t, \cdot)$ is a distributional solution of (2.7), classical computations show that u is indeed a weak solution of (1.4).
Conversely, suppose that u is a weak solution of (1.4), in the sense of Definition 2.1. We also consider

$$\begin{aligned} \bar{v}(t,dx) &:= \int_{\mathbb{R}} P(0,x_0,t,dx)u_0(dx_0) + \int_0^t ds \int_{\mathbb{R}^d} P(s,x_0,t,dx)u(s,dx_0)\Lambda(s,x_0,u(s,x_0),\nabla u(s,x_0))dx_0 \\ &= \int_{\mathbb{R}} P(0,x_0,t,dx)u_0(dx_0) + \int_0^t ds \int_{\mathbb{R}^d} P(s,x_0,t,dx)\bar{\Lambda}(u)(s,x_0)dx_0 \,, \end{aligned} \tag{2.14}$$

where $\bar{\Lambda}(u)(t,x) := u(s,x)\Lambda(s,x,u(s,x),\nabla u(s,x))$ for $(s,x) \in [0,T] \times \mathbb{R}^d$. We want to ensure that $u = \bar{v}$.
On the one hand, integrating the function $\bar{v}(t,\cdot)$ against a test function and using again that $P(s,x_0,t,\cdot)$ is a distributional solution of (2.7), we obtain that $\bar{v}$ is a weak solution of

$$\begin{cases} \partial_t \bar{v} = L_t^* \bar{v} + \bar{\Lambda}(u) \\ \bar{v}(0,\cdot) = u_0 \,, \end{cases} \tag{2.15}$$

On the other hand, u being a weak solution of (1.4), it also satisfies (2.15) (in the sense of distributions). We set $v := \bar{v} - u$. It follows that v and $\hat{v} := 0$ both satisfy (2.13). Uniqueness of the solution of (2.13) implies that $v = 0$, which concludes the proof. □

3 Feynman-Kac type representation

We suppose here the validity of Assumption 1. Let $u_0 \in \mathcal{P}(\mathbb{R}^d)$ and fix a random variable Y_0 distributed according to u_0 and consider the strong solution Y of (2.8).
The aim of this section is to show how a mild solution of (1.4) can be linked with a Feynman-Kac type equation, where we recall that a solution is given by a function $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ satisfying the second line equation of (1.2).

Given $\tilde{\Lambda} : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ a bounded, Borel measurable function, let us consider the measure-valued map $\mu : [0,T] \longrightarrow \mathcal{M}_f(\mathbb{R}^d)$ defined by

$$\int_{\mathbb{R}^d} \varphi(x)\mu(t,dx) = \mathbb{E}\Big[\varphi(Y_t)\exp\Big(\int_0^t \tilde{\Lambda}(s,Y_s)ds\Big)\Big], \text{ for all } \varphi \in \mathcal{C}_b(\mathbb{R}^d), t \in [0,T] \,. \tag{3.1}$$

The first proposition below shows how the map $t \mapsto \mu(t,\cdot)$ can be characterized as a solution of the linear parabolic PDE

$$\begin{cases} \partial_t v = L_t^* v + \tilde{\Lambda}(t,x)v \\ v(0,\cdot) = u_0 \,. \end{cases} \tag{3.2}$$

Before stating the corresponding proposition, we introduce the notion of *measure-mild solution*.

Definition 3.1. *A measure-valued map $\mu : [0,T] \to \mathcal{M}_f(\mathbb{R}^d)$ will be called* **measure-mild solution** *of* (3.2) *if for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $t \in [0,T]$,*

$$\int_{\mathbb{R}^d} \varphi(x)\mu(t,dx) = \int_{\mathbb{R}^d} \varphi(x) \int_{\mathbb{R}^d} u_0(dx_0) P(0,x_0,t,dx) + \int_{[0,t]\times\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} P(r,x_0,t,dx)\varphi(x) \Big) \tilde{\Lambda}(r,x_0)\mu(r,dx_0)dr. \tag{3.3}$$

Remark 3.2.

1. *Since μ is a (finite) measure valued function, by usual approximation arguments, it is not difficult to show that an equivalent formulation for Definition 2.1 can be expressed taking φ in $\mathcal{C}_b(\mathbb{R}^d)$ instead of $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$.*

2. *Although the definition of* **mild solution** *(see item 2. of Definition of 2.1) and the one of* **measure-mild solution** *seem to be formally close, the two concepts do not make sense in the same situations. Indeed, the notion of mild-solution makes sense for PDEs with nonlinear terms of the general form $\Lambda(t,x,u,\nabla u)$, whereas a measure-mild solution can exist only for linear PDEs. However, in the case where a measure μ on $\mathbb{R}^d$, absolutely continuous w.r.t. the Lebesgue measure dx, is a measure-mild solution of the linear PDE* (3.2)*, its density indeed coincides with the mild solution (in the sense of item 2. of Definition 2.1) of* (3.2).

Proposition 3.3. *Under Assumption 1 the measure-valued map μ defined by* (3.1) *is the unique measure-mild solution of*

$$\begin{cases} \partial_t v = L_t^* v + \tilde{\Lambda}(t,x)v \\ v(0,\cdot) = u_0 \ , \end{cases} \tag{3.4}$$

where the operator L_t^* is defined by (1.5).

Proof. We first prove that a function μ defined by (3.1) is a measure-mild solution of (3.4).
Observe that for all $t \in [0,T]$,

$$\exp\Big(\int_0^t \tilde{\Lambda}(r,Y_r)dr \Big) = 1 + \int_0^t \tilde{\Lambda}(r,Y_r) e^{\int_0^r \tilde{\Lambda}(s,Y_s)ds} dr \ . \tag{3.5}$$

From (3.1), it follows that for all test function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and $t \in [0,T]$,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x)\mu(t,dx) &= \mathbb{E}\Big[\varphi(Y_t)\exp\Big(\int_0^t \tilde{\Lambda}(r,Y_r)dr \Big)\Big] \\ &= \mathbb{E}[\varphi(Y_t)] + \int_0^t \mathbb{E}\Big[\varphi(Y_t)\tilde{\Lambda}(r,Y_r)e^{\int_0^r \tilde{\Lambda}(s,Y_s)ds}\Big] dr \ . \end{aligned} \tag{3.6}$$

On the one hand, by (2.9), we have

$$\mathbb{E}[\varphi(Y_t)] = \int_{\mathbb{R}^d} u_0(dx_0) \int_{\mathbb{R}^d} \varphi(x)P(0,x_0,t,dx) \ , \quad \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d) \text{ and } t \in [0,T] \ . \tag{3.7}$$

On the other hand, using (2.10) yields, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $0 \le r \le t$,

$$\begin{aligned} \mathbb{E}\Big[\varphi(Y_t)\tilde{\Lambda}(r,Y_r)e^{\int_0^r \tilde{\Lambda}(s,Y_s)ds}\Big] &= \mathbb{E}\Big[\tilde{\Lambda}(r,Y_r)e^{\int_0^r \tilde{\Lambda}(s,Y_s)ds}\mathbb{E}\Big[\varphi(Y_t)\Big|Y_r\Big]\Big] \\ &= \mathbb{E}\Big[\Big(\tilde{\Lambda}(r,Y_r)\int_{\mathbb{R}^d}\varphi(x)P(r,Y_r,t,dx)\Big)e^{\int_0^r \tilde{\Lambda}(s,Y_s)ds}\Big] \\ &= \int_{\mathbb{R}^d}\Big(\tilde{\Lambda}(r,x_0)\int_{\mathbb{R}^d}\varphi(x)P(r,x_0,t,dx)\Big)\mu(r,dx_0) \ , \end{aligned} \tag{3.8}$$

where the third equality above comes from (3.1) applied to the bounded, measurable test function $z \mapsto \tilde{\Lambda}(r,z)\int_{\mathbb{R}^d}\varphi(x)P(r,z,t,dx)$. Injecting (3.8) and (3.7) in the right-hand side (r.h.s.) of (3.6) gives for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $t \in [0,T]$,

$$\begin{aligned}\int_{\mathbb{R}^d}\varphi(x)\mu(t,dx) &= \int_{\mathbb{R}^d}u_0(dx_0)\int_{\mathbb{R}^d}P(0,x_0,t,dx)\varphi(x)dx \\ &\quad + \int_0^t\int_{\mathbb{R}^d}\mu(r,dx_0)\tilde{\Lambda}(r,x_0)\int_{\mathbb{R}^d}P(r,x_0,t,dx)\varphi(x)\, dr\ . \end{aligned} \tag{3.9}$$

It remains now to prove uniqueness of the measure-mild solution of (3.4).
We recall that $\mathcal{M}_f(\mathbb{R}^d)$ denotes the vector space of finite Borel measures on $\mathbb{R}^d$, that is here equipped with the total variation norm $\|\cdot\|_{TV}$. We also recall that an equivalent definition of the total variation norm is given by

$$\|\mu\|_{TV} = \sup_{\substack{\psi\in\mathcal{C}_b(\mathbb{R}^d)\\ \|\psi\|_\infty\le 1}}\Big|\int_{\mathbb{R}^d}\psi(x)\mu(dx)\Big|\ . \tag{3.10}$$

Consider $t \in [0,T]$ and let μ_1, μ_2 be two measure-mild solutions of PDE (3.4). We set $\nu := \mu_1 - \mu_2$. Since $\tilde{\Lambda}$ is bounded, we observe that (3.1) implies $\|\nu(t,\cdot)\|_{TV} < +\infty$. Moreover, taking into account item 1. of Remark 3.2, we have that ν satisfies,

$$\forall\, \varphi \in \mathcal{C}_b(\mathbb{R}^d),\ \int_{\mathbb{R}^d}\varphi(x)\nu(t,dx) = \int_0^t\int_{\mathbb{R}^d}\tilde{\Lambda}(r,x_0)\nu(r,dx_0)\int_{\mathbb{R}^d}\varphi(x)P(r,x_0,t,dx)\, dr\ . \tag{3.11}$$

Taking the supremum over φ such that $\|\varphi\|_\infty \le 1$ in each side of (3.11), we get

$$\|\nu(t,\cdot)\|_{TV} \quad\le\quad \sup_{(s,x)\in[0,T]\times\mathbb{R}^d}|\tilde{\Lambda}(s,x)|\int_0^t\|\nu(r,\cdot)\|_{TV}\, dr\ . \tag{3.12}$$

Gronwall's lemma implies that $\nu(t,\cdot) = 0$. Uniqueness of measure-mild solution for (3.4) follows. This ends the proof. □

The next lemma shows how a measure-mild solution of (3.4), which is a function defined on $[0,T]$ can be built by defining it recursively on each sub-interval of the form $[r, r+\tau]$. In particular, it will be used in Theorem 3.6 and Proposition 4.4. Its proof is postponed in Appendix (see Section 6.3).

Lemma 3.4. *Let us fix $\tau > 0$ be a real constant and $\delta := (\alpha_0 := 0 < \cdots < \alpha_k := k\tau < \cdots < \alpha_N := T)$ be a finite partition of $[0,T]$.*
A measure-valued map $\mu : [0,T] \to \mathcal{M}_f(\mathbb{R}^d)$ satisfies

$$\begin{cases}\mu(0,\cdot) = u_0 \\ \mu(t,dx) = \int_{\mathbb{R}^d}P(k\tau,x_0,t,dx)\mu(k\tau,dx_0) + \int_{k\tau}^t ds\int_{\mathbb{R}^d}P(s,x_0,t,dx)\tilde{\Lambda}(s,x_0)\mu(s,dx_0)\ ,\end{cases} \tag{3.13}$$

for all $t \in [k\tau,(k+1)\tau]$ and $k \in \{0,\cdots,N-1\}$, if and only if μ is a measure-mild solution (in the sense of Definition 3.1) of (3.4).

We now come back to the case where the bounded, Borel measurable real-valued function Λ is defined on $[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$. Let $u : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ belonging to $L^1([0,T],W^{1,1}(\mathbb{R}^d))$. In the sequel, we set $\tilde{\Lambda}^u(t,x) := \Lambda(t,x,u(t,x),\nabla u(t,x))$. μ^u will denote the measure-valued map μ defined by (3.1) with $\tilde{\Lambda} = \tilde{\Lambda}^u$, i.e.,

$$\int_{\mathbb{R}^d}\varphi(x)\mu^u(t,dx) = \mathbb{E}\Big[\varphi(Y_t)\exp\Big(\int_0^t\tilde{\Lambda}^u(s,Y_s)ds\Big)\Big],\ \text{for all } \varphi \in \mathcal{C}_b(\mathbb{R}^d), t \in [0,T]\ . \tag{3.14}$$

By Proposition 3.3, it follows that μ^u is the unique measure-mild solution of the linear PDE (3.4) with $\tilde{\Lambda} = \tilde{\Lambda}^u$. (3.14) can be interpreted as a Feynman-Kac type representation for the measure-mild solution μ^u of the linear PDE (3.4), for the corresponding $\tilde{\Lambda}^u$. More generally, Theorem 3.5 below establishes such representation formula for a mild solution of the semilinear PDE (1.4).

Theorem 3.5. *Assume that Assumption 1 is fulfilled. We indicate by Y the unique strong solution of* (2.8). *Suppose that $\Lambda : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is bounded and Borel measurable. A function $u : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ in $L^1([0,T], W^{1,1}(\mathbb{R}^d))$ is a mild solution of* (1.4) *if and only if, for all $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$, $t \in [0,T]$,*

$$\int_{\mathbb{R}^d} \varphi(x) u(t,x) dx = \mathbb{E}\Big[\varphi(Y_t) \exp\Big(\int_0^t \Lambda(s, Y_s, u(s,Y_s), \nabla u(s,Y_s))\Big)\Big] . \tag{3.15}$$

A function u verifying (3.15) will be called a **Feynman-Kac type representation** of (1.4).

Proof. We first suppose that u is a mild solution of (1.4). The aim is then to show that u satisfies the Feynman-Kac equation (3.15).
Since Λ is supposed to be bounded, Borel and u is Borel, it is clear that

$$\tilde{\Lambda}^u(t,x) := \Lambda(t,x,u(t,x), \nabla u(t,x)) , \tag{3.16}$$

is bounded and Borel measurable. On the one hand, by applying Proposition 3.3 with $\tilde{\Lambda}^u$, it follows that μ^u (defined by (3.14)) is the unique measure-mild solution of

$$\begin{cases} \partial_t v = L_t^* v + \tilde{\Lambda}^u(t,x) v \\ v(0,\cdot) = u_0 \, . \end{cases} \tag{3.17}$$

On the other hand, since u is supposed to be a mild solution of (1.4), $u(t,x)dx$ is also a measure-mild solution of (3.17). By uniqueness of the solution of (3.17), for all $t \in [0,T]$, we have $u(t,x)dx = \mu^u(t,dx)$. Taking into account item 1. of Remark 3.2, it implies for all $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$, $t \in [0,T]$,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) u(t,x) dx &= \int_{\mathbb{R}^d} \varphi(x) \mu^u(t,dx) \\ &= \mathbb{E}\Big[\varphi(Y_t) \exp\Big(\int_0^t \tilde{\Lambda}^u(s,Y_s)\Big) ds\Big], \text{ by (3.14)} \\ &= \mathbb{E}\Big[\varphi(Y_t) \exp\Big(\int_0^t \Lambda(s,Y_s,u(s,Y_s),\nabla u(s,Y_s)) ds\Big)\Big], \text{ by (3.16)} \; . \end{aligned}$$

Conversely, suppose that u satisfies the Feynman-Kac equation (3.15). Recalling (3.16) and setting $\bar{\mu}(t,dx) := u(t,x)dx$, (3.15) can be re-written

$$\int_{\mathbb{R}^d} \varphi(x) \bar{\mu}(t,dx) = \mathbb{E}\Big[\varphi(Y_t) \exp\Big(\int_0^t \tilde{\Lambda}^u(s,Y_s) ds\Big)\Big] , \quad \varphi \in \mathcal{C}_b(\mathbb{R}^d), t \in [0,T] \, . \tag{3.18}$$

Proposition 3.3 applied again with $\tilde{\Lambda}^u$ shows that $\bar{\mu}$ is the unique measure-mild solution of (3.17). In particular for every $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) u(t,x) dx &= \int_{\mathbb{R}^d} \varphi(x) \bar{\mu}(t,dx) \\ &= \int_{\mathbb{R}^d} u_0(dx_0) \int_{\mathbb{R}^d} P(0,x_0,t,dx) \varphi(x) dx \\ &+ \int_0^t \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} P(r,x_0,t,dx) \varphi(x) \Big) \Lambda(r,x_0,u(r,x_0),\nabla u(r,x_0)) u(r,x_0) dx_0 \; dr. \end{aligned} \tag{3.19}$$

This shows that u is a mild solution of (1.4) and concludes the proof. ∎

We now precise more restrictive assumptions to ensure regularity properties of the transition probability function $P(s, x_0, t, dx)$ used in the sequel.

Assumption 2. *1. Φ and g are functions defined on $[0,T] \times \mathbb{R}^d$ taking values respectively in $M_{d,p}(\mathbb{R})$ and $\mathbb{R}^d$. There exist $\alpha \in]0,1]$, $C_\alpha, L_\Phi, L_g > 0$, such that for any $(t, t', x, x') \in [0,T] \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$|\Phi(t,x) - \Phi(t,x')| \le C_\alpha |t-t'|^\alpha + L_\Phi |x - x'| \, ,$$
$$|g(t,x) - g(t',x')| \le C_\alpha |t-t'|^\alpha + L_g |x - x'| \, .$$

2. *Φ and g belong to $C_b^{0,3}$. In particular, Φ, g are uniformly bounded and M_Φ (resp. M_g) denote the upper bound of $|\Phi|$ (resp. $|g|$).*

3. *Φ is non-degenerate, i.e. there exists $c > 0$ such that for all $x \in \mathbb{R}^d$*

$$\inf_{s\in[0,T]} \inf_{v \in \mathbb{R}^d \setminus \{0\}} \frac{\langle v, \Phi(s,x)\Phi^t(s,x) v\rangle}{|v|^2} \ge c > 0. \tag{3.20}$$

4. *Λ is a Borel real-valued function defined on $[0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ and Lipschitz uniformly w.r.t. (t,x) i.e. there exists a finite positive real, L_Λ, such that for any $(t, x, z_1, z_1', z_2, z_2') \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^2 \times (\mathbb{R}^d)^2$, we have*

$$|\Lambda(t,x,z_1,z_2) - \Lambda(t,x,z_1',z_2')| \le L_\Lambda (|z_1 - z_1'| + |z_2 - z_2'|) \, . \tag{3.21}$$

5. *Λ is supposed to be uniformly bounded: let M_Λ be an upper bound for $|\Lambda|$.*

6. *u_0 is a Borel probability measure on $\mathbb{R}^d$ admitting a bounded density (still denoted by the same letter) belonging to $W^{1,1}(\mathbb{R}^d)$.*

Theorem 3.6. *Under Assumption 2, there exists a unique mild solution u of* (1.4) *in $L^1([0,T], W^{1,1}(\mathbb{R}^d)) \cap L^\infty([0,T] \times \mathbb{R}^d, \mathbb{R})$.*

The idea of the proof will be first to construct a unique "mild solution" u^k of (1.4) on each subintervals of the form $[k\tau, (k+1)\tau]$ with $k \in \{0, \cdots, N-1\}$ and $\tau > 0$ a constant supposed to be fixed for the moment. This will be the object of Lemma 3.7. Secondly we will show that the function $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, defined by being equal to u^k on each $[k\tau, (k+1)\tau]$, is indeed a mild solution of (1.4) on $[0,T] \times \mathbb{R}^d$. This will be a consequence of Lemma 3.4. Finally, uniqueness will follow classically from Lipschitz property of Λ.

Let us fix $\phi \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. For $r \in [0, T-\tau]$, we first define a function $\widehat{u_0}$ on $[r, r+\tau] \times \mathbb{R}^d$ by setting

$$\widehat{u_0}(r,\phi)(t,x) := \int_{\mathbb{R}^d} p(r, x_0, t, x)\phi(x_0) dx_0, \ (t,x) \in [r, r+\tau] \times \mathbb{R}^d. \tag{3.22}$$

Consider now the map $\Pi : L^1([r, r+\tau], W^{1,1}(\mathbb{R}^d)) \to L^1([r, r+\tau], W^{1,1}(\mathbb{R}^d))$ given by

$$\Pi(v)(t,x) := \int_r^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x) \Lambda(s, x_0, v + \widehat{u_0}(r,\phi), \nabla(v + \widehat{u_0}(r,\phi))) \big(v + \widehat{u_0}\big)(r,\phi)(s, x_0) dx_0 \, , \tag{3.23}$$

$$\Lambda(t, z, v, \nabla v) := \Lambda(t, z, v(t,z), \nabla v(t,z)) \quad \text{with} \quad (t,z) \in [0,T] \times \mathbb{R}^d \, , \tag{3.24}$$

that will also be used in the sequel.
Later, the dependence on r, ϕ will be omitted when it is self-explanatory. Since ϕ and u_0 belong to $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, we have

$$\|\widehat{u_0}(t,\cdot)\|_1 \le \|\phi\|_1 \quad \text{and} \quad \|\widehat{u_0}(t,\cdot)\|_\infty \le \|\phi\|_\infty, \ \text{if } t \in [r, r+\tau] \, . \tag{3.25}$$

The lemma below establishes, under a suitable choice of $\tau > 0$, existence and uniqueness of the mild solution on $[r, r+\tau]$, with initial condition ϕ at time r, i.e. existence and uniqueness of the fixed-point for the application Π.

Lemma 3.7. *Assume the validity of Assumption 2. Let $\phi \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.*
Let $M > 0$ such that $M \geq \max(\|\phi\|_\infty; \|\phi\|_1)$. Then, there is $\tau > 0$ only depending on M_Λ and on C_u, c_u (the constants coming from inequality (6.17), only depending on Φ, g) such that for any $r \in [0, T-\tau]$, Π admits a unique fixed-point in $L^1([r, r+\tau], B(0, M)) \cap B_\infty(0, M)$, where $B(0, M)$ (resp. $B_\infty(0, M)$) denotes the centered ball in $W^{1,1}(\mathbb{R}^d)$ (resp. $L^\infty([r, r+\tau] \times \mathbb{R}^d, \mathbb{R})$) of radius M.

Proof. We first insist on the fact that all along the proof, the dependence of $\widehat{u}_0$ w.r.t. r, ϕ in (3.23) will be omitted to simplify notations. Let us fix $r \in [0, T-\tau]$.
By item 1. of Lemma 6.4, the transition probabilities are absolutely continuous and $P(s, x_0, t, dx) = p(s, x_0, t, x)dx$ for some Borel function p. The rest of the proof relies on a fixed-point argument in the Banach space $L^1([r, r+\tau], W^{1,1}(\mathbb{R}^d))$ equipped with the norm $\|f\|_{1,1} := \int_r^{r+\tau} \|f(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds$ and for the map Π (3.23). Moreover, we emphasize that $L^1([r, r+\tau], B(0, M)) \cap B_\infty(0, M)$ is complete as a closed subset of $L^1([r, r+\tau], B(0, M))$.
We first check that $\Pi\Big(L^1([r, r+\tau], B(0, M)) \cap B_\infty(0, M)\Big) \subset L^1([r, r+\tau], B(0, M)) \cap B_\infty(0, M)$. Let us fix $v \in L^1([r, r+\tau], B(0, M)) \cap B_\infty(0, M)$. Indeed, for $t \in [r, r+\tau]$,

$$\begin{aligned} \|\Pi(v)(t, \cdot)\|_1 &= \int_{\mathbb{R}^d} |\Pi(v)(t, x)| dx \\ &\leq M_\Lambda \int_r^t \big(\|v(s, \cdot)\|_1 + \|\widehat{u_0}(s, \cdot)\|_1\big) ds \\ &\leq 2 M_\Lambda M \tau , \end{aligned} \tag{3.26}$$

where we have used the fact that $x \mapsto p(s, x_0, t, x)$ is a probability density, the boundedness of Λ and the bounds $\|v(s, \cdot)\|_1 \leq M$ and $\|\widehat{u_0}(s, \cdot)\|_1 \leq M$ for $s \in [r, r+\tau]$.
Let us fix $t \in [r, r+\tau]$. Since the transition probability function $x \mapsto p(s, x_0, t, x)$ is twice continuously differentiable for $0 \leq s < t \leq T$ (see item 2. of Lemma 6.4) and taking into account inequality (6.17), we differentiate under the integral sign to obtain that $\nabla \Pi(v)(t, \cdot)$ exists (in the sense of distributions) and is a real-valued function such that for almost all $x \in \mathbb{R}^d$,

$$\nabla \Pi(v)(t, x) = \int_r^t ds \int_{\mathbb{R}^d} \nabla_x p(s, x_0, t, x) \big(v + \widehat{u_0}\big)(s, x_0) \Lambda(s, x_0, v + \widehat{u_0}, \nabla(v + \widehat{u_0})) dx_0. \tag{3.27}$$

Integrating each side of (3.27) on $\mathbb{R}^d$ w.r.t. dx and using inequality (6.17) (with $(m_1, m_2) = (0, 1)$) yield

$$\begin{aligned} \|\nabla \Pi(v)(t, \cdot)\|_1 &= \int_{\mathbb{R}^d} |\nabla \Pi(v)(t, x)| dx \\ &\leq M_\Lambda \int_r^t \frac{ds}{\sqrt{t-s}} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} C_u \frac{e^{\frac{-c_u |x - x_0|}{t-s}}}{\sqrt{(t-s)^d}} \big(|v(s, x_0)| + |\widehat{u_0}(s, x_0)|\big) dx_0 \\ &= \hat{C} M_\Lambda \int_r^t \frac{ds}{\sqrt{t-s}} \int_{\mathbb{R}^d} \big(|v(s, x_0)| + |\widehat{u_0}(s, x_0)|\big) dx_0 \\ &\leq \hat{C} M_\Lambda \int_r^t \big(\|v(s, \cdot)\|_1 + \|\widehat{u_0}(s, \cdot)\|_1\big) \frac{ds}{\sqrt{t-s}} \\ &\leq 4 \hat{C} M_\Lambda M \sqrt{\tau} , \end{aligned} \tag{3.28}$$

with $\hat{C} := \hat{C}(C_u, c_u) > 0$ and C_u, c_u are the constants coming from inequality (6.17) and only depending on Φ and g. Consequently, taking into account (3.26) and (3.28), we obtain,

$$\|\Pi(v)\|_{1,1} = \int_r^{r+\tau} \|\Pi(v)(t,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} dt \quad \leq \quad 2MM_\Lambda(\tau^2 + 2\hat{C}\tau\sqrt{\tau}) \; . \tag{3.29}$$

Moreover using again inequality (6.17), with $(m_1, m_2) = (0, 0)$, gives existence of a constant $\bar{C} := \bar{C}(C_u, c_u)$ such that

$$\|\Pi(v)\|_\infty \quad \leq \quad 2\bar{C}MM_\Lambda\tau \; . \tag{3.30}$$

Now, setting

$$\tau := \min\Big(\sqrt{\frac{1}{6M_\Lambda}}; \Big(\frac{1}{12\hat{C}M_\Lambda}\Big)^{\frac{2}{3}}; \frac{1}{6\bar{C}M_\Lambda}\Big), \tag{3.31}$$

we have

$$2MM_\Lambda(\tau^2 + 2\hat{C}\tau\sqrt{\tau}) \leq \frac{2M}{3} \quad \text{and} \quad 2\bar{C}MM_\Lambda\tau \leq \frac{M}{3} \; ,$$

which implies

$$\|\Pi(v)\|_{1,1} \leq M \quad \text{and} \quad \|\Pi(v)\|_\infty \leq M \; .$$

We deduce that $\Pi(v) \in L^1([r, r+\tau], B(0, M)) \cap B_\infty(0, M)$.
Let us fix $t \in [r, r+\tau]$, $v_1, v_2 \in L^1([r, r+\tau], B(0, M)) \cap B_\infty(0, M)$. Λ being bounded and Lipschitz, the notation introduced in (3.24) and inequality (2.3) imply

$$\begin{aligned}
\|\Pi(v_1)(t,\cdot) - \Pi(v_2)(t,\cdot)\|_1 \quad \leq \quad & \int_r^t ds \int_{\mathbb{R}^d} \Big| v_1(s,x_0)\Lambda(s,x_0,v_1+\widehat{u_0},\nabla(v_1+\widehat{u_0})) - v_2(s,x_0)\Lambda(s,x_0,v_2+\widehat{u_0},\nabla(v_2+\widehat{u_0}))\Big| dx_0 \\
& + \int_r^t ds \int_{\mathbb{R}^d} |\widehat{u_0}(s,x_0)| \, \Big|\Lambda(s,x_0,v_1+\widehat{u_0},\nabla(v_1+\widehat{u_0})) - \Lambda(s,x_0,v_2+\widehat{u_0},\nabla(v_2+\widehat{u_0}))\Big| dx_0 \\
\leq \quad & \int_r^t ds \Big(\int_{\mathbb{R}^d} |v_1(s,x_0) - v_2(s,x_0)| \, |\Lambda(s,x_0,v_1+\widehat{u_0},\nabla(v_1+\widehat{u_0}))| dx_0 \\
& + L_\Lambda \int_r^t ds \int_{\mathbb{R}^d} \big(|\widehat{u_0}(s,x_0)| + |v_2(s,x_0)|\big) \, |v_1(s,x_0) - v_2(s,x_0)| dx_0 \\
& + L_\Lambda \int_r^t ds \int_{\mathbb{R}^d} \big(|\widehat{u_0}(s,x_0)| + |v_2(s,x_0)|\big) \, |\nabla v_1(s,x_0) - \nabla v_2(s,x_0)| dx_0 \\
\leq \quad & (M_\Lambda + 2ML_\Lambda) \int_r^t \|v_1(s,\cdot) - v_2(s,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds \; ,
\end{aligned} \tag{3.32}$$

where we have used the fact that $\int_{\mathbb{R}^d} p(s, x_0, t, x) dx = 1, \; 0 \leq s < t \leq T$.
In the same way and by using inequality (6.17) with $(m_1, m_2) = (0, 1)$,

$$\begin{aligned}
\Big\|\nabla\Big(\Pi(v_1) - \Pi(v_2)\Big)(t,\cdot)\Big\|_1 \quad \leq \quad & C_u(M_\Lambda + 2ML_\Lambda) \int_{\mathbb{R}^d} \int_r^t \int_{\mathbb{R}^d} \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{(t-s)^d}} e^{-c_u \frac{|x-x_0|^2}{t-s}} \Big(|v_1(s,x_0) - v_2(s,x_0)| \\
& + |\nabla v_1(s,x_0) - \nabla v_2(s,x_0)| \Big) dx_0 ds dx \; .
\end{aligned} \tag{3.33}$$

By Fubini's theorem we have

$$\Big\|\nabla\Big(\Pi(v_1) - \Pi(v_2)\Big)(t,\cdot)\Big\|_1 \quad \leq \quad \tilde{C} \int_r^t \frac{1}{\sqrt{t-s}} \|v_1(s,\cdot) - v_2(s,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds \; , \tag{3.34}$$

with $\tilde{C} := \tilde{C}(C_u, c_u, M_\Lambda, L_\Lambda, M)$ some positive constant. From (3.32) and (3.34), we deduce there exists a strictly positive constant $C = C(C_u, c_u, \Phi, g, \Lambda, M)$ (which may change from line to line) such that for all

$t \in [r, r+\tau]$,

$$\|\Pi(v_1)(t,\cdot) - \Pi(v_2)(t,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} \leq C\Big\{\int_r^t \|v_1(s,\cdot) - v_2(s,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds + \int_r^t \frac{1}{\sqrt{t-s}} \|v_1(s,\cdot) - v_2(s,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds\Big\}. \quad (3.35)$$

Iterating the procedure once again yields

$$\|\Pi^2(v_1)(t,\cdot) - \Pi^2(v_2)(t,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} \leq C\Big\{\int_r^t \int_r^s \|v_1(\theta,\cdot) - v_2(\theta,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} d\theta ds + \int_r^t \int_r^s \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s-\theta}} \|v_1(\theta,\cdot) - v_2(\theta,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} d\theta ds\Big\}, \quad (3.36)$$

for all $t \in [r, r+\tau]$. Interchanging the order in the second integral in the r.h.s. of (3.36), we obtain

$$\begin{aligned} \int_r^t \int_r^s \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s-\theta}} \|v_1(\theta,\cdot) - v_2(\theta,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} d\theta ds &= \int_r^t d\theta \, \|v_1(\theta,\cdot) - v_2(\theta,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} \int_\theta^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s-\theta}} ds \\ &= \int_r^t d\theta \, \|v_1(\theta,\cdot) - v_2(\theta,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} \int_0^\alpha \frac{1}{\sqrt{\alpha-\omega}} \frac{1}{\sqrt{\omega}} d\omega, \\ &\leq 4 \int_r^t \|v_1(\theta,\cdot) - v_2(\theta,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} d\theta, \end{aligned} \quad (3.37)$$

where the latter line above comes from the fact for all $\theta > 0$, $\int_0^\theta \frac{1}{\sqrt{\theta-\omega}} \frac{1}{\sqrt{\omega}} d\omega = \int_0^1 \frac{1}{\sqrt{1-\omega}} \frac{1}{\sqrt{\omega}} d\omega = \Gamma(\frac{1}{2})$, Γ denoting the Euler gamma function.
Injecting inequality (3.37) in (3.36), we obtain for all $t \in [r, r+\tau]$

$$\|\Pi^2(v_1)(t,\cdot) - \Pi^2(v_2)(t,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} \leq 5C \int_r^t \|v_1(s,\cdot) - v_2(s,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds. \quad (3.38)$$

Iterating previous inequality, one obtains the following. For all $k \geq 1$, $t \in [r, r+\tau]$,

$$\|\Pi^{2k}(v_1)(t,\cdot) - \Pi^{2k}(v_2)(t,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} \leq (5C)^k \int_r^t \frac{(t-s)^{k-1}}{(k-1)!} \|v_1(s,\cdot) - v_2(s,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds. \quad (3.39)$$

By induction on $k \geq 1$ (3.39) can indeed be established. Finally, by integrating each sides of (3.39) w.r.t. dt and using Fubini's theorem, for $k \geq 1$, we obtain

$$\int_r^{r+\tau} \|\Pi^{2k}(v_1)(t,\cdot) - \Pi^{2k}(v_2)(t,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} dt \leq (5C)^k \frac{T^{k-1}}{(k-1)!} \int_r^{r+\tau} \|v_1(s,\cdot) - v_2(s,\cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds. \quad (3.40)$$

For $k_0 \in \mathbb{N}$ large enough, $\frac{(5CT)^{k_0}}{T(k_0-1)!}$ will be strictly smaller than 1 and Π^{2k_0} will admit a unique fixed-point by Banach fixed-point theorem. In consequence, it implies easily that Π will also admit a unique fixed-point and this concludes the proof of Lemma 3.7. □

Proof of Theorem 3.6. Without restriction of generality, we can suppose there exists $N \in \mathbb{N}^*$ such that $T = N\tau$, where we recall that τ is given by (3.31). Similarly to the notations used in the preceding proof, in all the sequel, we agree that for $M > 0$, $B(0, M)$ (resp. $B_\infty(0, M)$) denotes the centered ball of radius M in $W^{1,1}(\mathbb{R}^d)$ (resp. in $L^\infty([0,T] \times \mathbb{R}^d, \mathbb{R})$ or in $L^\infty([r, r+\tau] \times \mathbb{R}^d, \mathbb{R})$ for $r \in [0, T-\tau]$ according to the context).

The notations introduced in (3.24) will also be used in the present proof.

Indeed, for $r = 0$, $\phi = u_0$ and $M \geq \max(\|u_0\|_\infty; \|u_0\|_1)$, Lemma 3.7 implies there exists a unique function $v^0 : [0, \tau] \times \mathbb{R}^d \to \mathbb{R}$ (belonging to $L^1([0, \tau], B(0, M)) \cap B_\infty(0, M)$) such that for $(t, x) \in [0, \tau] \times \mathbb{R}^d$,

$$v^0(t,x) = \int_0^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x)(v^0(s, x_0) + \widehat{u_0}^0(s, x_0))\Lambda\Big(s, x_0, v^0 + \widehat{u_0}^0, \nabla(v^0 + \widehat{u_0}^0)\Big)dx_0 \ , \tag{3.41}$$

where $\widehat{u_0}^0(t, x)$ is given by (3.22) with $\phi = u_0$, i.e.

$$\widehat{u_0}^0(t, x) = \int_{\mathbb{R}^d} p(0, x_0, t, x)u_0(x_0)dx_0 \ , \quad (t, x) \in [0, \tau] \times \mathbb{R}^d \ . \tag{3.42}$$

Setting $u^0 := \widehat{u_0}^0 + v^0$, i.e.

$$u^0(t, \cdot) = \int_{\mathbb{R}^d} p(0, x_0, t, \cdot)u_0(x_0)dx_0 + v^0(t, \cdot), \quad t \in [0, \tau] \ , \tag{3.43}$$

it appears that u^0 satisfies for all $(t, x) \in [0, \tau] \times \mathbb{R}^d$

$$u^0(t, x) = \int_{\mathbb{R}^d} p(0, x_0, t, x)u_0(x_0)dx_0 + \int_0^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x)u^0(s, x_0)\Lambda(s, x_0, u^0, \nabla u^0)dx_0 \ . \tag{3.44}$$

Let us fix $k \in \{1, \cdots, N-1\}$. Suppose now given a family of functions $u^1, u^2, \cdots, u^{k-1}$, where for all $j \in \{1, \cdots, k-1\}$, $u^j \in L^1([j\tau, (j+1)\tau], W^{1,1}(\mathbb{R}^d)) \cap L^\infty([j\tau, (j+1)\tau] \times \mathbb{R}^d, \mathbb{R})$ and satisfies for all $(t, x) \in [j\tau, (j+1)\tau] \times \mathbb{R}^d$,

$$u^j(t, x) = \int_{\mathbb{R}^d} p(j\tau, x_0, t, x)u^{j-1}(j\tau, x_0)dx_0 + \int_{j\tau}^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x)u^j(s, x_0)\Lambda(s, x_0, u^j, \nabla u^j)dx_0 \ . \tag{3.45}$$

Let us introduce

$$\begin{aligned} \widehat{u_0}^k(t, x) &:= \widehat{u_0}(u^{k-1})(t, x) \\ &= \int_{\mathbb{R}^d} p(k\tau, x_0, t, x)u^{k-1}(k\tau, x_0)dx_0 \ , \quad (t, x) \in [k\tau, (k+1)\tau] \times \mathbb{R}^d \ , \end{aligned} \tag{3.46}$$

where the second inequality comes from (3.22) with $r = k\tau$ and $\phi = u^{k-1}(k\tau, \cdot)$.
By choosing the real M large enough (i.e. $M \geq \max(\|u^{k-1}(k\tau, \cdot)\|_\infty; \|u^{k-1}(k\tau, \cdot)\|_1)$), Lemma 3.7 applied with $r = k\tau$, $\phi = u^{k-1}(k\tau, \cdot)$ implies existence and uniqueness of a function $v^k : [k\tau, (k+1)\tau] \times \mathbb{R}^d \to \mathbb{R}$ that belongs to $L^1([k\tau, (k+1)\tau], B(0, M)) \cap B_\infty(0, M)$ and satisfying

$$v^k(t, x) = \int_{k\tau}^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x)(v^k(s, x_0) + \widehat{u_0}^k(s, x_0))\Lambda\big(s, x_0, v^k + \widehat{u_0}^k, \nabla(v^k + \widehat{u_0}^k)\big)dx_0 \ , \tag{3.47}$$

for all $(t, x) \in [k\tau, (k+1)\tau] \times \mathbb{R}^d$. Setting $u^k := \widehat{u_0}^k + v^k$, we have for all $(t, x) \in [k\tau, (k+1)\tau] \times \mathbb{R}^d$

$$u^k(t, x) = \int_{\mathbb{R}^d} p(k\tau, x_0, t, x)u^{k-1}(k\tau, x_0)dx_0 + \int_{k\tau}^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x)u^k(s, x_0)\Lambda(s, x_0, u^k, \nabla u^k)dx_0 \ . \tag{3.48}$$

Consequently, by induction we can construct a family of functions $(u^k : [k\tau, (k+1)\tau] \times \mathbb{R}^d \to \mathbb{R})_{k=0,\cdots,N-1}$ such that for all $k \in \{0, \cdots, N-1\}$, $u^k \in L^1([k\tau, (k+1)\tau], W^{1,1}(\mathbb{R}^d)) \cap L^\infty([k\tau, (k+1)\tau] \times \mathbb{R}^d, \mathbb{R})$ and verifies (3.48).

We now consider the real-valued function $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ defined as being equal to u^k (defined by (3.48)) on each interval $[k\tau, (k+1)\tau]$. Then, Lemma 3.4 applied with τ given by (3.31) and

$$\delta = (\alpha_0 := 0 < \cdots < \alpha_k := k\tau < \cdots < \alpha_N := T = N\tau) \quad , \quad \mu(t, dx) = u(t, x)dx, \tag{3.49}$$

shows that u is a mild solution of (1.4) on $[0,T]\times\mathbb{R}^d$, in the sense of Definition 2.1, item 2. It now remains to ensure that u is indeed the unique mild solution of (1.4) on $[0,T]\times\mathbb{R}^d$ belonging to $L^1([0,T],W^{1,1}(\mathbb{R}^d))\cap L^\infty([0,T]\times\mathbb{R}^d,\mathbb{R})$. This follows, in a classical way, by boundedness and Lipschitz property of Λ.
Indeed, if U, V are two mild solutions of (1.4), then very similar computations as the ones done in (3.32), (3.34), and (3.39) to obtain (3.40) give the following. There exists $C := C(\Phi,g,\Lambda,U,V) > 0$ such that

$$\int_0^T \|U(t,\cdot)-V(t,\cdot)\|_{W^{1,1}(\mathbb{R}^d)}dt \le (5C)^j\frac{T^{j-1}}{(j-1)!}\int_0^T \|U(s,\cdot)-V(s,\cdot)\|_{W^{1,1}(\mathbb{R}^d)}ds. \tag{3.50}$$

If we choose $j\in\mathbb{N}^\star$ large enough so that $(5C)^j\frac{T^{j-1}}{(j-1)!} < 1$, we obtain $U(t,x) = V(t,x)$ for almost all $(t,x)\in[0,T]\times\mathbb{R}^d$. This concludes the proof of Theorem 3.6. □

Corollary 3.8. *Under Assumption 2, there exists a unique function $u : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}$ satisfying the Feynman-Kac equation* (3.15)*. In particular, such u coincides with the mild solution of* (1.4).

In the case where the function Λ does not depend on ∇u, existence and uniqueness of a solution of (1.4) in the mild sense can be proved under weaker assumptions. This is the object of the following result.

Theorem 3.9. *Assume that Assumption 1 is satisfied. Let $u_0\in\mathcal{P}(\mathbb{R}^d)$ admitting a bounded density (still denoted by the same letter). Let Y the the strong solution of* (2.8) *with prescribed Y_0.*
We suppose that the transition probability function P (see (2.7)*) admits a density p such that $P(s,x_0,t,dx) = p(s,x_0,t,x)dx$, for all $s,t\in[0,T]$, $x_0\in\mathbb{R}^d$. Λ is supposed to satisfy items 4. and 5. of Assumption 2. Then, there exists a unique mild solution u of* (1.4) *in $L^1([0,T],L^1(\mathbb{R}^d))$, i.e. u satisfies*

$$u(t,x) = \int_{\mathbb{R}^d} p(0,x_0,t,x)u_0(x_0)dx_0 + \int_0^t\int_{\mathbb{R}^d} p(s,x_0,t,x)u(s,x_0)\Lambda(s,x_0,u(s,x_0))dx_0\,ds, \quad (t,x)\in[0,T]\times\mathbb{R}^d\,. \tag{3.51}$$

Proof. Since this theorem can be proved in a very similar way as Theorem 3.6 but with simpler computations, we omit the details. □

4 Existence/uniqueness of the Regularized Feynman-Kac equation

In this section, we introduce a regularized version of PDE (1.4) to which we associate a regularized Feynman-Kac equation corresponding to a regularized version of (3.15). This regularization procedure constitutes a preliminary step for the construction of a particle scheme approximating (3.15). Indeed, as detailed in the next section devoted to the particle approximation, the point dependence of Λ on u and ∇u will require to derive from a discrete measure (based on the particle system) estimates of densities u and their derivatives ∇u, which can classically be achieved by kernel convolution.
Assumption 1 is in force. Let u_0 be a Borel probability measure on $\mathbb{R}^d$ and Y_0 a random variable distributed according to u_0. We consider Y the strong solution of the SDE (2.8).

Let us consider $(K_\varepsilon)_{\varepsilon>0}$, a sequence of mollifiers such that

$$K_\varepsilon \xrightarrow[\varepsilon\to 0]{} \delta_0, \text{ (weakly)} \quad\text{and}\quad \forall\,\varepsilon>0, K_\varepsilon\in W^{1,1}(\mathbb{R}^d)\cap W^{1,\infty}(\mathbb{R}^d)\,. \tag{4.1}$$

Let $\Lambda : [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ be bounded, Borel measurable. As announced, we introduce the following integro-PDE corresponding to a regularized version of (1.4)

$$\begin{cases} \partial_t\gamma_t = L_t^*\gamma_t + \gamma_t\Lambda(t,x,K_\varepsilon*\gamma_t,\nabla K_\varepsilon*\gamma_t) \\ \gamma_0 = u_0\,. \end{cases} \tag{4.2}$$

The concept of mild solution associated to this type of equation is clarified by the following definition.

Definition 4.1. *A Borel measure-valued function $\gamma : [0,T] \longrightarrow \mathcal{M}_f(\mathbb{R}^d)$ will be called a* **mild solution** *of* (4.2) *if it satisfies, for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $t \in [0,T]$,*

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(x)\gamma(t,dx) &= \int_{\mathbb{R}^d} \varphi(x) \int_{\mathbb{R}^d} u_0(dx_0) P(0,x_0,t,dx) \\ &\quad + \int_{[0,t]\times\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \varphi(x) P(s,x_0,t,dx) \Big) \Lambda\big(s,x_0,(K_\varepsilon * \gamma(s,\cdot))(x_0),(\nabla K_\varepsilon * \gamma(s,\cdot))(x_0)\big)\gamma(s,dx_0)ds \,.\end{aligned} \tag{4.3}$$

Similarly as Theorem 3.5, we straightforwardly derive the following equivalence result.

Proposition 4.2. *Suppose that Assumption 1 and* (4.1) *are fulfilled. We indicate by Y the unique strong solution of* (2.8)*. A Borel measure-valued function $\gamma^\varepsilon : [0,T] \longrightarrow \mathcal{M}_f(\mathbb{R}^d)$ is a mild solution of* (4.2) *if and only if, for all $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$, $t \in [0,T]$,*

$$\int_{\mathbb{R}^d} \varphi(x)\gamma_t^\varepsilon(dx) = \mathbb{E}\Big[\varphi(Y_t)\exp\Big(\int_0^t \Lambda\big(s,Y_s,(K_\varepsilon * \gamma_s^\varepsilon)(Y_s),(\nabla K_\varepsilon * \gamma_s^\varepsilon)(Y_s)\big)\, ds\Big)\Big] \,. \tag{4.4}$$

Proof. The proof follows the same lines as the proof of Theorem 3.5. First assume that γ^ε satisfies (4.4), we can show that γ^ε is a mild solution (4.2) by imitating the first step of the proof of Proposition 3.3. Secondly, the converse is proved by applying Proposition 3.3 with $\tilde{\Lambda}(t,x) := \Lambda(t,x,(K_\varepsilon * \gamma_t^\varepsilon)(x),(\nabla K_\varepsilon * \gamma_t^\varepsilon)(x))$ and $\mu(t,dx) := \gamma_t^\varepsilon(dx)$. □

Let us now prove existence and uniqueness of a mild solution for the integro-PDE (4.2). To this end, we proceed similarly as for the proof of Theorem 3.6 using Lemma 3.7. Let $\tau > 0$ be a constant supposed to be fixed for the moment and let us fix $\varepsilon > 0$, $r \in [0, T-\tau]$. $\mathcal{B}([r,r+\tau],\mathcal{M}_f(\mathbb{R}^d))$ denotes the space of bounded, measure-valued maps, where $\mathcal{M}_f(\mathbb{R}^d)$ is equipped with the total variation norm $\|\cdot\|_{TV}$. We introduce the measure-valued application $\Pi_\varepsilon : \beta \in \mathcal{B}([r,r+\tau],\mathcal{M}_f(\mathbb{R}^d)) \longrightarrow \Pi_\varepsilon(\beta)$, defined by

$$\begin{aligned}\Pi_\varepsilon(\beta)(t,dx) &= \int_r^t \int_{\mathbb{R}^d} P(s,x_0,t,dx)\Lambda\big(s,x_0,(K_\varepsilon * \widehat{\beta}(s,\cdot))(x_0),(\nabla K_\varepsilon * \widehat{\beta}(s,\cdot))(x_0)\big)\,\widehat{\beta}(s,dx_0)ds \,. \\ \widehat{\beta}(s,\cdot) &= \beta(s,\cdot) + \widehat{u_0}(s,\cdot) \,,\end{aligned} \tag{4.5}$$

where the function $\widehat{u_0}$, defined on $[r,r+\tau] \times \mathcal{M}_f(\mathbb{R}^d)$, is given by

$$\widehat{u_0}(r,\pi)(t,dx) := \int_{\mathbb{R}^d} p(r,x_0,t,dx)\pi(dx_0), \; t \in [r,r+\tau], \; \pi \in \mathcal{M}_f(\mathbb{R}^d) \,, \tag{4.6}$$

similarly to (3.22). In the sequel, the dependence of $\widehat{u}_0$ w.r.t. r,π will be omitted when it is self-explanatory.

Lemma 4.3. *Assume the validity of items 4. and 5. of Assumption 2 and of* (4.1)*. Let $\pi \in B(0,M)$.*
Let us fix $\varepsilon > 0$ and $M > 0$ such that $M \geq \|\pi\|_{TV}$. Then, there is $\tau > 0$ only depending on M_Λ such that for any $r \in [0,T-\tau]$, Π_ε admits a unique fixed-point in $\mathcal{B}([r,r+\tau],B(0,M))$, where $B(0,M)$ denotes here the centered ball in $(\mathcal{M}_f(\mathbb{R}^d),\|\cdot\|_{TV})$ with radius M.

Proof. Let us define $\tau := \frac{1}{2M_\Lambda}$. For every $\lambda \geq 0$, $\mathcal{B}([r,r+\tau],\mathcal{M}_f(\mathbb{R}^d))$ will be equipped with one of the equivalent norms

$$\|\beta\|_{TV,\lambda} := \sup_{t\in[r,r+\tau]} e^{-\lambda t}\|\beta(t,\cdot)\|_{TV} \,. \tag{4.7}$$

Recalling (4.5), where $\widehat{u_0}$ is defined by (4.6), it follows that for all $\beta \in \mathcal{B}([r, r+\tau], B(0,M))$, $t \in [r, r+\tau]$,

$$\|\Pi_\varepsilon(\beta)(t,\cdot)\|_{TV} \leq M_\Lambda \int_r^t \|\beta(s,\cdot)\|_{TV} ds + M_\Lambda M \tau \leq 2MM_\Lambda \tau \leq M \; , \tag{4.8}$$

where for the latter inequality of (4.8) we have used the definition of $\tau := \frac{1}{2M_\Lambda}$. We deduce that $\Pi(\mathcal{B}([r, r+\tau], B(0,M))) \subset \mathcal{B}([r, r+\tau], B(0,M))$.
Consider now $\beta^1, \beta^2 \in \mathcal{B}([r, r+\tau], B(0,M))$. For all $\lambda > 0$ we have

$$\begin{aligned}
\|\Pi_\varepsilon(\beta^1(t,\cdot)) - \Pi_\varepsilon(\beta^2(t,\cdot))\|_{TV} &\leq \int_0^t \|\beta^1(s,\cdot) - \beta^2(s,\cdot)\|_{TV} (L_\Lambda \|K_\varepsilon\|_\infty \|\beta^1(s,\cdot)\|_{TV} + M_\Lambda) ds \\
&\quad + L_\Lambda \|\nabla K_\varepsilon\|_\infty \int_r^t \|\beta^1(s,\cdot)\|_{TV} \|\beta^1(s,\cdot) - \beta^2(s,\cdot)\|_{TV} ds \\
&\quad + L_\Lambda (\|K_\varepsilon\|_\infty + \|\nabla K_\varepsilon\|_\infty) \int_r^t \|\widehat{u_0}(s,\cdot)\|_{TV} \|\beta^1(s,\cdot) - \beta^2(s,\cdot)\|_{TV} ds \\
&\leq C_{\varepsilon,T} \int_0^t \|\beta^1(s,\cdot) - \beta^2(s,\cdot)\|_{TV} ds \\
&\leq C_{\varepsilon,T} \int_0^t e^{s\lambda} \, \|\beta^1 - \beta^2\|_{TV,\lambda} \, ds \\
&= C_{\varepsilon,T} \|\beta^1 - \beta^2\|_{TV,\lambda} \, \frac{e^{\lambda t} - 1}{\lambda} \; ,
\end{aligned} \tag{4.9}$$

with $C_{\varepsilon,T} := 2L_\Lambda M(\|K_\varepsilon\|_\infty + \|\nabla K_\varepsilon\|_\infty) + M_\Lambda$. It follows

$$\begin{aligned}
\|\Pi_\varepsilon(\beta^1) - \Pi_\varepsilon(\beta^2)\|_{TV,\lambda} &= \sup_{t \in [r, r+\tau]} e^{-\lambda t} \|\Pi(\beta^1)(t,\cdot) - \Pi(\beta^2)(t,\cdot)\|_{TV} \\
&\leq C_{\varepsilon,T} \|\beta^1 - \beta^2\|_{TV,\lambda} \sup_{t \geq 0} \Big(\frac{1 - e^{-\lambda t}}{\lambda} \Big) \\
&\leq \frac{C_{\varepsilon,T}}{\lambda} \|\beta^1 - \beta^2\|_{TV,\lambda} \, .
\end{aligned} \tag{4.10}$$

Hence, taking $\lambda > C_{\varepsilon,T}$, Π_ε is a contraction on $\mathcal{B}([r, r+\tau], B(0,M))$.
Since $\mathcal{B}\big([r, r+\tau], (\mathcal{M}_f(\mathbb{R}^d), \|\cdot\|_{TV,\lambda})\big)$ is a Banach space whose $\mathcal{B}([r, r+\tau], B(0,M))$ is a closed subset, the proof ends by a simple application of Banach fixed-point theorem. □

The next step is to show how the proposition above, with the help of Lemma 3.4, permits us to construct a mild solution of (4.2). The reasoning is similar to the one explained in the proof of Theorem 3.6. Indeed, without restriction of generality, we can suppose there exists $N \in \mathbb{N}^\star$ such that $T = N\tau$. Then, for all $k = 0, \cdots, N-1$, Lemma 4.3 applied on each interval $[k\tau, (k+1)\tau]$ (with $r = k\tau$, $\pi = \beta_\varepsilon^{k-1}(k\tau, \cdot)$ for $k \geq 1$ and $\pi = u_0$ for $k = 0$) gives existence of a family of measure-valued maps $(\beta_\varepsilon^k : [k\tau, (k+1)\tau] \to \mathcal{M}_f(\mathbb{R}^d), k = 0, \cdots, N-1)$ defined by

$$\begin{aligned}
\beta_\varepsilon^k(t, dx) &= \int_{k\tau}^t \int_{\mathbb{R}^d} P(s, x_0, t, dx) \Lambda\big(s, x_0, (K_\varepsilon * \widehat{\beta}_\varepsilon^k(s,\cdot))(x_0), (\nabla K_\varepsilon * \widehat{\beta}_\varepsilon^k(s,\cdot))(x_0)\big) \, \widehat{\beta}_\varepsilon^k(s, dx_0) ds \, . \\
\widehat{\beta_\varepsilon^k}(s,\cdot) &= \beta_\varepsilon^k(s,\cdot) + \widehat{u_0}^k(s,\cdot) \; ,
\end{aligned} \tag{4.11}$$

where for $k = 0$, $t \in [0, \tau]$,

$$\widehat{u}_0^0(t, dx) = \int_{\mathbb{R}^d} P(0, x_0, t, dx) u_0(dx_0) \; , \quad \text{by (4.6) with } \pi = u_0 \; , \tag{4.12}$$

and for all $k \in \{1, \cdots, N\}$, $t \in [k\tau, (k+1)\tau]$,

$$\begin{aligned} \widehat{u_0}^k(t, dx) &:= \widehat{u_0}(\beta_\varepsilon^{k-1})(t, dx) \\ &= \int_{\mathbb{R}^d} P(k\tau, x_0, t, dx)\beta_\varepsilon^{k-1}(k\tau, dx_0)\ , \quad \text{by (4.6) with } \pi = \beta_\varepsilon^{k-1}(k\tau, \cdot)\ . \end{aligned} \tag{4.13}$$

We now consider the following measure-valued maps $\widehat{U_0} : [0,T] \to \mathcal{M}_f(\mathbb{R}^d)$ and $\beta_\varepsilon : [0,T] \to \mathcal{M}_f(\mathbb{R}^d)$ defined by their restrictions on each interval $[k\tau, (k+1)\tau], k = 0, \cdots, N-1$ such that

$$\widehat{U_0}(t,x) := \widehat{u_0}^k(t,x) \quad \text{and} \quad \beta_\varepsilon(t,x) := \beta_\varepsilon^k(t,x) \quad \text{for } (t,x) \in [k\tau, (k+1)\tau] \times \mathbb{R}^d\ , \tag{4.14}$$

and we finally define $\gamma^\varepsilon : [0,T] \to \mathcal{M}_f(\mathbb{R}^d)$ by

$$\gamma^\varepsilon := \widehat{U_0} + \beta_\varepsilon \text{ on } [0,T] \times \mathbb{R}^d. \tag{4.15}$$

To ensure that γ^ε is indeed a mild solution on $[0,T] \times \mathbb{R}^d$ (in the sense of Definition 4.1) of the integro-PDE (4.2), it is enough to apply Lemma 3.4 with $\tau := \frac{1}{2M_\Lambda}$, $\mu(t,dx) := \gamma^\varepsilon(t,dx)$ and $(\alpha_k := k\tau)_{k=0,\cdots,N}$.

Previous discussion leads us to the following proposition.

Proposition 4.4. *Suppose the validity of Assumption 1 and items 4. and 5. of Assumption 2. Suppose also that* (4.1) *is fulfilled. Let us fix $\varepsilon > 0$ and let γ^ε denote the map defined by* (4.15)*. The following statements hold.*

1. *γ^ε is the unique mild solution of the integro-PDE* (4.2)*, see Definition 4.1.*
2. *γ^ε is the unique solution to the regularized Feynman-Kac equation* (4.4)*.*

Proof. The existence of a mild solution γ^ε of (4.2) has already been proved through the discussion just above. It remains to justify uniqueness. Consider $\gamma^{\varepsilon,1}$, $\gamma^{\varepsilon,2}$ be two mild solutions of (4.3). Then, with similar computations as the ones leading to inequality (4.10), there exists a constant $\mathfrak{C} := \mathfrak{C}(M_\Lambda, L_\Lambda, \|K_\varepsilon\|_\infty, \|\nabla K_\varepsilon\|_\infty) > 0$ such that

$$\|\gamma^{\varepsilon,1} - \gamma^{\varepsilon,2}\|_{TV,\lambda} \leq \frac{\mathfrak{C}}{\lambda}\|\gamma^{\varepsilon,1} - \gamma^{\varepsilon,2}\|_{TV,\lambda}\ , \tag{4.16}$$

for all $\lambda > 0$ and where we recall that $\|\cdot\|_{TV,\lambda}$ has been defined by (4.7). Taking $\lambda > \mathfrak{C}$, uniqueness follows. This shows item 1. Item 2. follows then by Proposition 4.2. □

The theorem below states the convergence of the solution of the regularized Feynman-Kac equation (4.4) to the solution to the Feynman-Kac equation (3.15). This is equivalent to the convergence of the solution of the regularized PDE (4.2) to solution of the target PDE (1.4), when the regularization parameter ε goes to zero.

Theorem 4.5. *Suppose the validity of Assumption 2. Suppose also that* (4.1) *is fulfilled. For any $\varepsilon > 0$, consider the real valued function u^ε such that for any $t \in [0,T]$,*

$$u^\varepsilon(t,\cdot) := K_\varepsilon * \gamma_t^\varepsilon\ , \tag{4.17}$$

where γ^ε is the unique solution of both (4.4) *and* (4.2)*. Then u^ε converges to u, the unique solution of both* (3.15) *and* (1.4)*, in the sense that*

$$\|u^\varepsilon(t,\cdot) - u(t,\cdot)\|_1 + \|\nabla u^\varepsilon(t,\cdot) - \nabla u(t,\cdot)\|_1 \xrightarrow[\varepsilon \to 0]{} 0\ , \quad \textit{for any } t \in [0,T]\ . \tag{4.18}$$

Before proving Theorem 4.5, we state and prove a preliminary lemma.

Lemma 4.6. *Suppose the validity of Assumption 2 and of* (4.1)*. Consider u the unique solution of* (3.15)*, then for all $t \in [0,T]$*

$$u(t,x) = F_0(t,x) + \int_0^t \mathbb{E}\Big[p(s,Y_s,t,x)\Lambda(s,Y_s,u,\nabla u)e^{\int_0^s \Lambda(r,Y_r,u,\nabla u)dr}\Big]ds,\ dx \text{ a.e.} \tag{4.19}$$

For a given $\varepsilon > 0$, consider u^ε defined by (4.17)*. Then for almost all $x \in \mathbb{R}^d$ and all $t \in [0,T]$,*

$$u^\varepsilon(t,x) = (K_\varepsilon * F_0(t,\cdot))(x) + \int_0^t \mathbb{E}\Big[(K_\varepsilon * p(s,Y_s,t,\cdot))(x)\Lambda(s,Y_s,u^\varepsilon,\nabla u^\varepsilon)e^{\int_0^s \Lambda(r,Y_r,u^\varepsilon,\nabla u^\varepsilon)dr}\Big]ds\ , \tag{4.20}$$

where $F_0(t,x) := \int_{\mathbb{R}^d} p(0,x_0,t,x)u_0(x_0)dx_0$ for $t > 0, x \in \mathbb{R}^d$ and $F_0(0,\cdot) := u_0$. We remark that we have used again the notation

$$\Lambda(s,\cdot,v,\nabla v) := \Lambda(s,\cdot,v(s,\cdot),\nabla v(s,\cdot)), \quad t \in [0,T]\ , \tag{4.21}$$

for $v \in L^1([0,T],W^{1,1}(\mathbb{R}^d))$.

Proof. Equalities (4.19) and (4.20) are proved in a very similar way, so we only provide the proof of equation (4.20).
We observe that for all $t \in [0,T]$, $v \in L^1([0,T],W^{1,1}(\mathbb{R}^d))$,

$$e^{\int_0^t \Lambda(r,Y_r,v(r,Y_r),\nabla v(r,Y_r))dr} \quad = \quad 1 + \int_0^t \Lambda(r,Y_r,v(r,Y_r),\nabla v(r,Y_r))e^{\int_0^r \Lambda(s,Y_s,v(s,Y_s),\nabla v(s,Y_s))ds}\ . \tag{4.22}$$

Taking into account the notation introduced in (4.21), (4.22) above implies for almost all $x \in \mathbb{R}^d$,

$$\begin{aligned}
u^\varepsilon(t,x) &= \mathbb{E}\Big[K_\varepsilon(x-Y_t)\ \exp\Big\{\int_0^t \Lambda\Big(s,Y_s,u^\varepsilon,\nabla u^\varepsilon\Big)ds\Big\}\Big] \\
&= \mathbb{E}\Big[K_\varepsilon(x-Y_t)\Big] + \int_0^t \mathbb{E}\Big[K_\varepsilon(x-Y_t)\Lambda(r,Y_r,u^\varepsilon,\nabla u^\varepsilon)e^{\int_0^r \Lambda(s,Y_s,u^\varepsilon,\nabla u^\varepsilon)ds}\Big]dr \\
&= \int_{\mathbb{R}^d} K_\varepsilon(x-y)\int_{\mathbb{R}^d} p(0,x_0,t,y)u_0(x_0)dx_0\ dy + \\
&\quad \int_0^t \mathbb{E}\Big[\mathbb{E}\Big[K_\varepsilon(x-Y_t)\Big|Y_r\Big]\Lambda(r,Y_r,u^\varepsilon,\nabla u^\varepsilon)e^{\int_0^r \Lambda(s,Y_s,u^\varepsilon,\nabla u^\varepsilon)ds}\Big]dr \\
&= (K_\varepsilon * F_0)(t,\cdot)(x) + \\
&\quad \int_0^t \mathbb{E}\Big[\Big(\int_{\mathbb{R}^d} K_\varepsilon(x-y)p(r,Y_r,t,y)dy\Big)\ \Lambda(r,Y_r,u^\varepsilon,\nabla u^\varepsilon)e^{\int_0^r \Lambda(s,Y_s,u^\varepsilon,\nabla u^\varepsilon)ds}\Big]dr \\
&= (K_\varepsilon * F_0)(t,\cdot)(x) + \\
&\quad \int_0^t \mathbb{E}\Big[(K_\varepsilon * p(r,Y_r,t,\cdot))(x)\ \Lambda(r,Y_r,u^\varepsilon,\nabla u^\varepsilon)e^{\int_0^r \Lambda(s,Y_s,u^\varepsilon,\nabla u^\varepsilon)ds}\Big]dr\ .
\end{aligned} \tag{4.23}$$

This ends the proof. □

Proof of Theorem 4.5. In this proof, C denotes a real constant that may change from line to line, only depending on M_Λ, L_Λ, C_u, c_u and $\|u_0\|_\infty$, where we recall that the constants C_u, c_u come from inequality (6.17) (and only depend on Φ, g).
We first observe that for $t = 0$, the convergence of $u^\varepsilon(0,\cdot)$ (resp. $\nabla u^\varepsilon(0,\cdot)$) to $u(0,\cdot)$ (resp. $\nabla u(0,\cdot)$) in $L^1(\mathbb{R}^d)$-norm when ε goes to 0 is clear. Let us fix $t \in (0,T]$.

By Lemma 4.6, for almost all $x \in \mathbb{R}^d$, we have the decomposition

$$
\begin{aligned}
u^\varepsilon(t,x) - u(t,x) &= (K_\varepsilon * F_0(t,\cdot))(x) - F_0(t,x) + \\
&\int_0^t \mathbb{E}\Big[\Big\{(K_\varepsilon * p(s,Y_s,t,\cdot))(x) - p(s,Y_s,t,x)\Big\}\Lambda(s,Y_s,u^\varepsilon,\nabla u^\varepsilon)e^{\int_0^s \Lambda(r,Y_r,u^\varepsilon,\nabla u^\varepsilon)dr}\Big]ds + \\
&\int_0^t \mathbb{E}\Big[p(s,Y_s,t,x)\Big\{\Lambda(s,Y_s,u^\varepsilon,\nabla u^\varepsilon)e^{\int_0^s \Lambda(r,Y_r,u^\varepsilon,\nabla u^\varepsilon)dr} - \Lambda(s,Y_s,u,\nabla u)e^{\int_0^s \Lambda(r,Y_r,u,\nabla u)dr}\Big\}\Big]ds\,.
\end{aligned}
\tag{4.24}
$$

By integrating the absolute value of both sides of (4.24) w.r.t. dx, it follows there exists a constant $C > 0$ such that

$$
\begin{aligned}
\|u^\varepsilon(t,\cdot) - u(t,\cdot)\|_1 \quad &\leq \quad C\Big\{\|K_\varepsilon * F_0 - F_0\|_1 + \int_0^t \mathbb{E}\Big[\|K_\varepsilon * p(s,Y_s,t,\cdot) - p(s,Y_s,t,\cdot)\|_1\Big]ds \\
&+ \int_0^t \mathbb{E}\Big[|u^\varepsilon(s,Y_s) - u(s,Y_s)| + |\nabla u^\varepsilon(s,Y_s) - \nabla u(s,Y_s)|\Big]ds \\
&+ \int_0^t \int_0^s \mathbb{E}\Big[|u^\varepsilon(r,Y_r) - u(r,Y_r)| + |\nabla u^\varepsilon(r,Y_r) - \nabla u(r,Y_r)|\Big]dr\ ds\Big\}\,.
\end{aligned}
\tag{4.25}
$$

Moreover, denoting p_s the law density of Y_s, by inequality (6.18) of Lemma 6.4 we get

$$
\begin{aligned}
\mathbb{E}\Big[|u^\varepsilon(s,Y_s) - u(s,Y_s)|\Big] &= \int_{\mathbb{R}^d} |u^\varepsilon(s,x) - u(s,x)|p_s(x)dx \\
&\leq C\ \|u_0\|_\infty \int_{\mathbb{R}^d} |u^\varepsilon(s,x) - u(s,x)|dx \\
&= C\ \|u^\varepsilon(s,\cdot) - u(s,\cdot)\|_1\,, \quad s \in [0,T]\,,
\end{aligned}
\tag{4.26}
$$

and

$$
\begin{aligned}
\mathbb{E}\Big[|\nabla u^\varepsilon(s,Y_s) - \nabla u(s,Y_s)|\Big] &= \int_{\mathbb{R}^d} |\nabla u^\varepsilon(s,x) - \nabla u(s,x)|p_s(x)dx \\
&\leq C\ \|u_0\|_\infty \int_{\mathbb{R}^d} |\nabla u^\varepsilon(s,x) - \nabla u(s,x)|dx \\
&= C\ \|\nabla u^\varepsilon(s,\cdot) - \nabla u(s,\cdot)\|_1\,, \quad s \in [0,T]\,.
\end{aligned}
\tag{4.27}
$$

Injecting (4.26) and (4.27) into the r.h.s. of (4.25), it comes

$$
\begin{aligned}
\|u^\varepsilon(t,\cdot) - u(t,\cdot)\|_1 \quad &\leq \quad C\Big\{\|K_\varepsilon * F_0 - F_0\|_1 + \int_0^t \mathbb{E}\Big[\|K_\varepsilon * p(s,Y_s,t,\cdot) - p(s,Y_s,t,\cdot)\|_1\Big]ds \\
&+ \int_0^t \|u^\varepsilon(s,\cdot) - u(s,\cdot)\|_1 + \|\nabla u^\varepsilon(s,\cdot) - \nabla u(s,\cdot)\|_1 ds\Big\}\,.
\end{aligned}
\tag{4.28}
$$

Now, we need to bound $\|\nabla u^\varepsilon(t,\cdot) - \nabla u(t,\cdot)\|_1$. To this end, we can remark that for almost all $x \in \mathbb{R}^d$,

$$
\nabla u(t,x) = \nabla F_0(t,x) + \int_0^t \mathbb{E}\Big[\nabla_x p(s,Y_s,t,x)\Lambda(s,Y_s,u,\nabla u)e^{\int_0^s \Lambda(r,Y_r,u,\nabla u)dr}\Big]ds\,,
\tag{4.29}
$$

and

$$
\begin{aligned}
\nabla u^\varepsilon(t,x) \quad &= \quad (K_\varepsilon * \nabla F_0(t,\cdot))(x) \\
&+ \int_0^t \mathbb{E}\Big[(K_\varepsilon * \nabla_x p(s,Y_s,t,\cdot))(x)\Lambda(s,Y_s,u^\varepsilon,\nabla u^\varepsilon)e^{\int_0^s \Lambda(r,Y_r,u^\varepsilon,\nabla u^\varepsilon)dr}\Big]ds\,.
\end{aligned}
\tag{4.30}
$$

These equalities follow by computing the derivative of $u(t,\cdot)$ and $u^\varepsilon(t,\cdot)$ in the sense of distributions. Taking into account (4.29) and (4.30), it is easy to see that very similar arguments as those invoked above to prove (4.28), lead to

$$\begin{aligned} \|\nabla u^\varepsilon(t,\cdot)-\nabla u(t,\cdot)\|_1 \quad \le \quad & C\Big\{\|K_\varepsilon * \nabla F_0(t,\cdot)-\nabla F_0(t,\cdot)\|_1 + \int_0^t \mathbb{E}\Big[\|K_\varepsilon * \nabla_x p(s,Y_s,t,\cdot)-\nabla_x p(s,Y_s,t,\cdot)\|_1\Big] ds \\ & + \int_0^t \|u^\varepsilon(s,\cdot)-u(s,\cdot)\|_1 + \|\nabla u^\varepsilon(s,\cdot)-\nabla u(s,\cdot)\|_1 ds\Big\}\,. \end{aligned} \tag{4.31}$$

Gathering (4.28) together with (4.31) yields

$$\begin{aligned} \|u^\varepsilon(t,\cdot)-u(t,\cdot)\|_1 + \|\nabla u^\varepsilon(t,\cdot)-\nabla u(t,\cdot)\|_1 \quad \le \quad & C\,\Big\{\|K_\varepsilon * F_0(t,\cdot)-F_0(t,\cdot)\|_1 + \|K_\varepsilon * \nabla F_0(t,\cdot)-\nabla F_0(t,\cdot)\|_1 + \\ & \int_0^t \mathbb{E}\Big[\|K_\varepsilon * p(s,Y_s,t,\cdot)-p(s,Y_s,t,\cdot)\|_1\Big] ds + \\ & \int_0^t \mathbb{E}\Big[\|K_\varepsilon * \nabla_x p(s,Y_s,t,\cdot)-\nabla_x p(s,Y_s,t,\cdot)\|_1\Big] ds\Big\} \\ & + \int_0^t \|u^\varepsilon(s,\cdot)-u(s,\cdot)\|_1 + \|\nabla u^\varepsilon(s,\cdot)-\nabla u(s,\cdot)\|_1 ds\,. \end{aligned} \tag{4.32}$$

Applying Gronwall's lemma to the real-valued function

$$t \mapsto \|u^\varepsilon(t,\cdot)-u(t,\cdot)\|_1 + \|\nabla u^\varepsilon(t,\cdot)-\nabla u(t,\cdot)\|_1\;,$$

we obtain

$$\begin{aligned} \|u^\varepsilon(t,\cdot)-u(t,\cdot)\|_1 + \|\nabla u^\varepsilon(t,\cdot)-\nabla u(t,\cdot)\|_1 \quad \le \quad & Ce^{CT}\,\Big\{\|K_\varepsilon * F_0(t,\cdot)-F_0(t,\cdot)\|_1 + \|K_\varepsilon * \nabla F_0(t,\cdot)-\nabla F_0(t,\cdot)\|_1 + \\ & \int_0^t \mathbb{E}\Big[\|K_\varepsilon * p(s,Y_s,t,\cdot)-p(s,Y_s,t,\cdot)\|_1\Big] ds + \\ & \int_0^t \mathbb{E}\Big[\|K_\varepsilon * \nabla_x p(s,Y_s,t,\cdot)-\nabla_x p(s,Y_s,t,\cdot)\|_1\Big] ds\Big\}. \end{aligned} \tag{4.33}$$

Since $F_0(t,\cdot)$, $\nabla F_0(t,\cdot)$, $x \mapsto p(s,x_0,t,x)$ and $x \mapsto \nabla_x p(s,x_0,t,x)$ belong to $L^1(\mathbb{R}^d)$, classical properties of convergence of the mollifiers give

$$\|K_\varepsilon * F_0(t,\cdot)-F_0(t,\cdot)\|_1 \to 0, \quad \|K_\varepsilon * \nabla F_0(t,\cdot)-\nabla F_0(t,\cdot)\|_1 \to 0, \tag{4.34}$$

and

$$\|K_\varepsilon * p(s,Y_s,t,\cdot)-p(s,Y_s,t,\cdot)\|_1 \to 0, \quad \|K_\varepsilon * \nabla_x p(s,Y_s,t,\cdot)-\nabla_x p(s,Y_s,t,\cdot)\|_1 \to 0,\ a.s.\,. \tag{4.35}$$

Moreover, by inequality (6.17) of Lemma 6.4, there exists a constant $C := C(C_u, c_u) > 0$ such that for $0 \le s < t \le T$,

$$\|K_\varepsilon * p(s,Y_s,t,\cdot)-p(s,Y_s,t,\cdot)\|_1 + \|K_\varepsilon * \nabla_x p(s,Y_s,t,\cdot)-\nabla_x p(s,Y_s,t,\cdot)\|_1 \le 2C(1+\frac{1}{\sqrt{t-s}})\ a.s.\,. \tag{4.36}$$

Lebesgue dominated convergence theorem then implies that the third and fourth terms in the r.h.s. of (4.33) converge to 0 when ε goes to 0. This ends the proof. □

Proposition 4.7. *We assume here that* $(K_\varepsilon)_{\varepsilon>0}$ *is explicitly given by*

$$K_\varepsilon(x) := \frac{1}{\varepsilon^d} K(\frac{x}{\varepsilon}) \,, \tag{4.37}$$

with $K \geq 0$ *satisfying*

$$\int_{\mathbb{R}^d} K(x)\,dx = 1 \,, \quad \int_{\mathbb{R}^d} x\,K(x)\,dx = 0 \quad \textit{and} \quad \kappa := \frac{1}{2}\int_{\mathbb{R}^d} |x|\,K(x)\,dx < \infty. \tag{4.38}$$

Let u^ε *be the real-valued function defined by* (4.17)*, (such that for any* $t \in [0,T]$*,* $u^\varepsilon(t,\cdot) := K_\varepsilon * \gamma_t^\varepsilon$*) with* K_ε *given by* (4.37)*. Under Assumption 2 and in the particular case where the function* $\Lambda(t,x,u)$ *does not depend on the gradient* ∇u*, there exists a constant* $C := C(\kappa, C_u, c_u) > 0$ *such that, for all* $t \in (0,T]$*,*

$$\|u^\varepsilon(t,\cdot) - u(t,\cdot)\|_1 \leq \ \varepsilon C\big(\tfrac{1}{\sqrt{t}} + 2\sqrt{t}\big), \tag{4.39}$$

with C_u*,* c_u *denoting the constants given by* (6.17) *(only depending on* Φ*,* g*).*

Proof. This proof is based on the same arguments as the ones used in the proof of Theorem 4.5 since in the present case, Λ only depends on (t,x,u) and not on ∇u. In particular, we obtain for $t \in]0,T]$,

$$\|u^\varepsilon(t,\cdot) - u(t,\cdot)\|_1 \leq Ce^{CT}\|K_\varepsilon * F_0(t,\cdot) - F_0(t,\cdot)\|_1 + \int_0^t \mathbb{E}\Big[\|K_\varepsilon * p(s,Y_s,t,\cdot) - p(s,Y_s,t,\cdot)\|_1\Big]ds, \tag{4.40}$$

that corresponds to inequality (4.33) in the proof above, without the term containing the gradient ∇u. Invoking inequality (6.17) of Lemma 6.4 and inequality (6.9) of Lemma 6.3 with $H = K$, and successively with $f = F_0(t,\cdot)$ and $f = p(s,y,t,\cdot)$ for fixed $y \in \mathbb{R}^d$, imply that

$$\|K_\varepsilon * F_0(t,\cdot) - F_0(t,\cdot)\|_1 \leq \frac{\varepsilon\kappa\mathfrak{C}}{\sqrt{t}} \,, \quad 0 < t \leq T, \tag{4.41}$$

and

$$\|K_\varepsilon * p(s,Y_s,t,\cdot) - p(s,Y_s,t,\cdot)\|_1 \leq \frac{\varepsilon\kappa\mathfrak{C}}{\sqrt{t-s}} \text{ , a.e,} \quad 0 \leq s < t < T \,, \tag{4.42}$$

with $\mathfrak{C} = \mathfrak{C}(C_u, c_u)$. This concludes the proof of (4.39). □

5 Particles system algorithm

To simplify notations in the rest of the paper, f_t will denote $f(t)$ where $f : [0,T] \to E$ is an E-valued Borel function and (E, d_E) a metric space.
In previous sections, we have studied existence, uniqueness for a semilinear PDE of the form (1.4) and we have established a Feynman-Kac type representation for the corresponding solution u, see Theorem 3.5. The regularized form of (1.4) is the integro-PDE (4.2) for which we have established well-posedness in Proposition 4.4 1. In the sequel, we denote by γ^ε the corresponding solution and again by $u^\varepsilon(t,x) := (K_\varepsilon * \gamma_t^\varepsilon)(x)$ (see (4.17)). We recall that u^ε converges to u, when the regularization parameter ε vanishes to 0, see Theorem 4.5. In the present section, we propose a Monte Carlo approximation $u^{\varepsilon,N}$ of u^ε, providing an original numerical approximation of the semilinear PDE (1.4), when both the number of particles $N \to \infty$ and the regularization parameter $\varepsilon \to 0$ slowly enough. Let u_0 be a Borel probability measure on $\mathcal{P}(\mathbb{R}^d)$.

5.1 Convergence of the particle system

We suppose the validity of Assumption 2.

For fixed $N \in \mathbb{N}^\star$, let $(W^i)_{i=1,\cdots,N}$ be a family of independent Brownian motions and $(Y_0^i)_{i=1,\cdots,N}$ be i.i.d. random variables distributed according to $u_0(x)dx$. For any $\varepsilon > 0$, we define the measure-valued functions $(\gamma_t^{\varepsilon,N})_{t\in[0,T]}$ such that for any $t \in [0,T]$

$$\begin{cases} \xi_t^i = \xi_0^i + \int_0^t \Phi(s,\xi_s^i)dW_s^i + \int_0^t g(s,\xi_s^i)ds\ , \quad \text{for } i = 1,\cdots,N\ , \\ \xi_0^i = Y_0^i \quad \text{for } i = 1,\cdots,N\ , \\ \gamma_t^{\varepsilon,N} = \dfrac{1}{N}\displaystyle\sum_{i=1}^N V_t\big(\xi^i,(K_\varepsilon * \gamma^{\varepsilon,N})(\xi^i),(\nabla K_\varepsilon * \gamma^{\varepsilon,N})(\xi^i)\big)\delta_{\xi_t^i},\ . \end{cases} \tag{5.1}$$

where $(K_\varepsilon)_{\varepsilon>0}$ denotes a sequence of mollifiers such that for all $\varepsilon > 0$, $K_\varepsilon \in W^{1,1}(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$ and we recall that V_t is given by (2.1). The first line of (5.1) is a d-dimensional classical SDE whose strong existence and pathwise uniqueness are ensured by classical theorems for Lipschitz coefficients. Moreover $\xi^i, i = 1,\cdots,N$ are i.i.d. In the following lemma, we prove by a fixed-point argument that the third line equation of (5.1) has a unique solution.

Lemma 5.1. *We suppose the validity of Assumption 2. Let us fix $\varepsilon > 0$ and $N \in \mathbb{N}^\star$. Consider the i.i.d. system $(\xi^i)_{i=1,\cdots,N}$ of particles, solution of the two first equations of* (5.1)*. Then, there exists a unique function $\gamma^{\varepsilon,N}$: $[0,T] \to \mathcal{M}_f(\mathbb{R}^d)$ such that for all $t \in [0,T]$, $\gamma_t^{\varepsilon,N}$ is solution of* (5.1)*.*

Proof. The proof relies on a fixed-point argument applied to the map $T^{\varepsilon,N} : \mathcal{C}([0,T],\mathcal{M}_f(\mathbb{R}^d)) \to \mathcal{C}([0,T],\mathcal{M}_f(\mathbb{R}^d))$ given by

$$T^{\varepsilon,N}(\gamma)(t) : \gamma \mapsto \frac{1}{N}\sum_{i=1}^N V_t\big(\xi^i,(K_\varepsilon * \gamma)(\xi^i),(\nabla K_\varepsilon * \gamma)(\xi^i)\big)\delta_{\xi_t^i}\ . \tag{5.2}$$

In the rest of the proof, the notation $T_t^{\varepsilon,N}(\gamma)$ will denote $T^{\varepsilon,N}(\gamma)(t)$.

In order to apply the Banach fixed-point theorem, we emphasize that $\mathcal{C}\big([0,T],\mathcal{M}_f(\mathbb{R}^d)\big)$ is equipped with one of the equivalent norms $\|\cdot\|_{TV,\lambda}$, $\lambda \geq 0$, defined by

$$\|\gamma\|_{TV,\lambda} := \sup_{t\in[0,T]} e^{-\lambda t}\|\gamma(t,\cdot)\|_{TV}\ , \tag{5.3}$$

and for which $\big(\mathcal{C}([0,T],\mathcal{M}_f(\mathbb{R}^d)),\|\cdot\|_{TV,\lambda}\big)$ is still complete.

From now on, it remains to ensure that $T^{\varepsilon,N}$ is indeed a contraction with respect $\|\gamma\|_{TV,\lambda}$ for some λ. To simplify notations, we set for all $i \in \{1,\cdots,N\}$,

$$T_t^{\varepsilon,N,i}(\gamma) := V_t\big(\xi^i,(K_\varepsilon * \gamma)(\xi^i),(\nabla K_\varepsilon * \gamma)(\xi^i)\big)\ , \quad (t,\gamma) \in [0,T] \times \mathcal{C}\big([0,T],\mathcal{M}_f(\mathbb{R}^d)\big)\ , \tag{5.4}$$

to re-write (5.2) in the form

$$T_t^{\varepsilon,N}(\gamma) = \frac{1}{N}\sum_{i=1}^N T_t^{\varepsilon,N,i}(\gamma)\delta_{\xi_t^i}\ , \quad (t,\gamma) \in [0,T] \times \mathcal{C}\big([0,T],\mathcal{M}_f(\mathbb{R}^d)\big)\ . \tag{5.5}$$

Let $\lambda > 0$. Consider now $\gamma^1,\gamma^2 \in \mathcal{C}\big([0,T],\mathcal{M}_f(\mathbb{R}^d)\big)$. On the one hand, taking into account (2.1) and (2.3), for all $t \in [0,T]$, $i \in \{1,\cdots,N\}$, we have

$$\begin{aligned}
|T_t^{\varepsilon,N,i}(\gamma^1) - T_t^{\varepsilon,N,i}(\gamma^2)| &\leq L_\Lambda e^{TM_\Lambda}\int_0^t \Big(|(K_\varepsilon * \gamma^1)(\xi_s^i) - (K_\varepsilon * \gamma^2)(\xi_s^i)| \\
&\qquad + |(\nabla K_\varepsilon * \gamma^1)(\xi_s^i) - (\nabla K_\varepsilon * \gamma^2)(\xi_s^i)\Big) ds \\
&\leq C\int_0^t \|\gamma_s^1 - \gamma_s^2\|_{TV} ds \\
&\leq C\int_0^t e^{s\lambda}\, \|\gamma^1 - \gamma^2\|_{TV,\lambda}\, ds \\
&= C\|\gamma^1 - \gamma^2\|_{TV,\lambda}\, \frac{e^{\lambda t}-1}{\lambda} \ ,
\end{aligned} \tag{5.6}$$

with $C = C(T, \|K_\varepsilon\|_\infty, \|\nabla K_\varepsilon\|_\infty, L_\Lambda, M_\Lambda)$. It follows that

$$\begin{aligned}
\|T^{\varepsilon,N}(\gamma^1) - T^{\varepsilon,N}(\gamma^2)\|_{TV,\lambda} &\leq \frac{1}{N}\sum_{i=1}^N \|T^{\varepsilon,N,i}(\gamma^1)\delta_{\xi^i} - T^{\varepsilon,N,i}(\gamma^2)\delta_{\xi^i}\|_{TV,\lambda} \\
&\leq \frac{C}{\lambda}\|\gamma^1 - \gamma^2\|_{TV,\lambda} \ .
\end{aligned} \tag{5.7}$$

By taking $\lambda > C$ and invoking Banach fixed-point theorem, we end the proof. □

After the preceding preliminary considerations, we can state and prove the main result of the section.

Proposition 5.2. *We suppose the validity of Assumption 2. Assume that the kernel K is a probability density verifying*

$$K \in W^{1,1}(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)\ , \quad \int_{\mathbb{R}^d} |x|^{d+1}\, K(x)dx < \infty\ , \quad \textit{and} \quad \int_{\mathbb{R}^d} |x|^{d+1}\, |\nabla K(x)|dx < \infty\ . \tag{5.8}$$

For $\varepsilon > 0$, we suppose that K_ε is explicitly given by (4.37) *with K satisfying* (5.8)*. Let u^ε be the real valued function defined by* (4.17)*, and $u^{\varepsilon,N}$ such that for any $t \in [0,T]$,*

$$u^{\varepsilon,N}(t,\cdot) := K_\varepsilon * \gamma_t^{\varepsilon,N}\ , \tag{5.9}$$

where $\gamma^{\varepsilon,N}$ is defined by the third line of (5.1)*. There is a constant C (only depending on M_Φ, M_g, M_Λ, $\|K\|_\infty$, $\|\nabla K\|_\infty$, L_Φ, L_g, L_Λ, T) such that the following holds. For all $t \in [0,T]$ and $N \in \mathbb{N}^*$, $\varepsilon > 0$ verifying min$(N, N\varepsilon^d) > C$ we have*

$$\mathbb{E}\Big[\|u_t^{\varepsilon,N} - u_t^\varepsilon\|_1\Big] + \mathbb{E}\Big[\|\nabla u_t^{\varepsilon,N} - \nabla u_t^\varepsilon\|_1\Big] \leq \frac{C}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}}\ . \tag{5.10}$$

Proof. Let us fix $\varepsilon > 0$, $N \in \mathbb{N}^\star$. For any $\ell = 1,\cdots,d$, we introduce the real-valued function G_ε^ℓ defined on $\mathbb{R}^d$ such that

$$G_\varepsilon^\ell(x) := \frac{1}{\varepsilon^d}\frac{\partial K}{\partial x_\ell}\left(\frac{x}{\varepsilon}\right)\ , \quad \text{for almost all } x \in \mathbb{R}^d\ . \tag{5.11}$$

By (5.8), there exists a finite positive constant C independent of ε such that $\|G_\varepsilon^\ell\|_\infty \leq \frac{C}{\varepsilon^d}$ and $\|G_\varepsilon^\ell\|_1 = \|G_1^\ell\|_1 \leq C$. In the sequel, C will always denote a finite positive constant independent of (ε, N) that may change from line to line. For any $t \in [0,T]$, we introduce the random Borel measure $\tilde\gamma_t^{\varepsilon,N}$ on $\mathbb{R}^d$, defined by

$$\tilde\gamma_t^{\varepsilon,N} := \frac{1}{N}\sum_{i=1}^N V_t\big(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)\big)\, \delta_{\xi_t^i}\ . \tag{5.12}$$

One can first decompose the error on the l.h.s of inequality (5.10) as follows

$$\begin{aligned}\mathbb{E}\Big[\|u_t^{\varepsilon,N}-u_t^{\varepsilon}\|_1\Big]+\mathbb{E}\Big[\|\nabla u_t^{\varepsilon,N}-\nabla u_t^{\varepsilon}\|_1\Big] &= \mathbb{E}\Big[\|K_\varepsilon*(\gamma_t^{\varepsilon,N}-\gamma_t^{\varepsilon})\|_1\Big]+\frac{1}{\varepsilon}\sum_{\ell=1}^{d}\mathbb{E}\Big[\|G_\varepsilon^{\ell}*(\gamma_t^{\varepsilon,N}-\gamma_t^{\varepsilon})\|_1\Big]\\ &\le \mathbb{E}\Big[\|K_\varepsilon*(\gamma_t^{\varepsilon,N}-\tilde\gamma_t^{\varepsilon,N})\|_1\Big]+\frac{1}{\varepsilon}\sum_{\ell=1}^{d}\mathbb{E}\Big[\|G_\varepsilon^{\ell}*(\gamma_t^{\varepsilon,N}-\tilde\gamma_t^{\varepsilon,N})\|_1\Big]\\ &\quad +\mathbb{E}\Big[\|K_\varepsilon*(\tilde\gamma_t^{\varepsilon,N}-\gamma_t^{\varepsilon})\|_1\Big]+\frac{1}{\varepsilon}\sum_{\ell=1}^{d}\mathbb{E}\Big[\|G_\varepsilon^{\ell}*(\tilde\gamma_t^{\varepsilon,N}-\gamma_t^{\varepsilon})\|_1\Big]\\ &= \mathbb{E}\Big[\|A_t^{\varepsilon,N}\|_1\Big]+\mathbb{E}\Big[\|A_t'^{\varepsilon,N}\|_1\Big]+\mathbb{E}\Big[\|B_t^{\varepsilon,N}\|_1\Big]+\mathbb{E}\Big[\|B_t'^{\varepsilon,N}\|_1\Big]\,,\end{aligned} \tag{5.13}$$

where, for all $t\in[0,T]$,

$$\begin{cases} A_t^{\varepsilon,N}(x) &:= \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}K_\varepsilon(x-\xi_t^i)\Big[V_t\big(\xi^i,u^{\varepsilon,N}(\xi^i),\nabla u^{\varepsilon,N}(\xi^i)\big)-V_t\big(\xi^i,u^{\varepsilon}(\xi^i),\nabla u^{\varepsilon}(\xi^i)\big)\Big]\\ A_t'^{\varepsilon,N}(x) &:= \dfrac{1}{\varepsilon}\displaystyle\sum_{\ell=1}^{d}\Bigg|\frac{1}{N}\sum_{i=1}^{N}G_\varepsilon^{\ell}(x-\xi_t^i)\Big[V_t\big(\xi^i,u^{\varepsilon,N}(\xi^i),\nabla u^{\varepsilon,N}(\xi^i)\big)-V_t\big(\xi^i,u^{\varepsilon}(\xi^i),\nabla u^{\varepsilon}(\xi^i)\big)\Big]\Bigg|\\ B_t^{\varepsilon,N}(x) &:= \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}K_\varepsilon(x-\xi_t^i)V_t\big(\xi^i,u^{\varepsilon}(\xi^i),\nabla u^{\varepsilon}(\xi^i)\big)-\mathbb{E}\Big[K_\varepsilon(x-\xi_t^1)V_t\big(\xi^1,u^{\varepsilon}(\xi^1),\nabla u^{\varepsilon}(\xi^1)\big)\Big]\\ B_t'^{\varepsilon,N}(x) &:= \dfrac{1}{\varepsilon}\displaystyle\sum_{\ell=1}^{d}\Bigg|\frac{1}{N}\sum_{i=1}^{N}G_\varepsilon^{\ell}(x-\xi_t^i)V_t\big(\xi^i,u^{\varepsilon}(\xi^i),\nabla u^{\varepsilon}(\xi^i)\big)-\mathbb{E}\Big[G_\varepsilon^{\ell}(x-\xi_t^1)V_t\big(\xi^1,u^{\varepsilon}(\xi^1),\nabla u^{\varepsilon}(\xi^1)\big)\Big]\Bigg|\,.\end{cases} \tag{5.14}$$

We will proceed in two steps, first bounding $\mathbb{E}\Big[\|B_t^{\varepsilon,N}\|_1\Big]$ and $\mathbb{E}\Big[\|B_t'^{\varepsilon,N}\|_1\Big]$ and then $\mathbb{E}\Big[\|A_t^{\varepsilon,N}\|_1\Big]$ and $\mathbb{E}\Big[\|A_t'^{\varepsilon,N}\|_1\Big]$.

Step 1: Bounding $\mathbb{E}\|B_t^{\varepsilon,N}\|_1$ and $\mathbb{E}\|B_t'^{\varepsilon,N}\|_1$. For any $i\in\{1,\cdots,N\}$ and $(t,x)\in[0,T]\times\mathbb{R}^d$ we set

$$P_i^{\varepsilon}(t,x):=K_\varepsilon(x-\xi_t^i)V_t\big(\xi^i,u^{\varepsilon}(\xi^i),\nabla u^{\varepsilon}(\xi^i)\big)-\mathbb{E}\Big[K_\varepsilon(x-\xi_t^i)V_t\big(\xi^i,u^{\varepsilon}(\xi^i),\nabla u^{\varepsilon}(\xi^i)\big)\Big]\,. \tag{5.15}$$

Notice that for fixed $(t,x)\in[0,T]\times\mathbb{R}^d$, $(P_i^{\varepsilon}(t,x))_{i=1,\cdots,N}$ are i.i.d. centered square integrable random variables. Hence using Cauchy-Schwarz inequality, we have

$$\begin{aligned}\mathbb{E}\Big[\|B_t^{\varepsilon,N}\|_1\Big] &= \int_{\mathbb{R}^d}\mathbb{E}\Big[\Big|\frac{1}{N}\sum_{i=1}^{N}P_i^{\varepsilon}(t,x)\Big|\Big]dx\\ &\le \int_{\mathbb{R}^d}\sqrt{\mathbb{E}\Big[\Big(\frac{1}{N}\sum_{i=1}^{N}(P_i^{\varepsilon}(t,x))\Big)^2\Big]}dx\\ &= \frac{1}{\sqrt N}\int_{\mathbb{R}^d}\sqrt{\mathbb{E}\Big[(P_1^{\varepsilon}(t,x))^2\Big]}dx\,.\end{aligned} \tag{5.16}$$

By the boundedness assumption on Λ (item 5. of Assumption 2.,

$$\mathbb{E}[(P_1^{\varepsilon}(t,x))^2]\le 4e^{2M_\Lambda T}\mathbb{E}[(K_\varepsilon(x-\xi_t^1))^2]\,,$$

which implies

$$\mathbb{E}\Big[\|B_t^{\varepsilon,N}\|_1\Big]\le\frac{C}{\sqrt N}\int_{\mathbb{R}^d}\sqrt{\mathbb{E}\Big[(K_\varepsilon(x-\xi_t^1))^2\Big]}dx=\frac{C}{\sqrt N}\frac{\sqrt{\int_{\mathbb{R}^d}K^2(x)dx}}{\sqrt{\varepsilon^d}}\int_{\mathbb{R}^d}\sqrt{H_\varepsilon*p_t}(x)dx\,, \tag{5.17}$$

where p_t is the law density of ξ_t^1 and H_ε is the probability density on $\mathbb{R}^d$ such that for almost all $x \in \mathbb{R}^d$, $H_\varepsilon(x) := \frac{1}{\int_{\mathbb{R}^d} K^2(x)dx} \frac{1}{\varepsilon^d} K^2(\frac{x}{\varepsilon})$, which is well-defined thanks to assumption (5.8). Finally, applying Lemma 6.2 with $G = \frac{K^2}{\|K\|_2^2}$ and $f = p_t$ we obtain

$$\mathbb{E}\Big[\|B_t^{\varepsilon,N}\|_1\Big] \leq \frac{C}{\sqrt{N\varepsilon^d}}\ , \quad \text{for } \varepsilon \text{ small enough.} \tag{5.18}$$

Proceeding similarly for the term $B_t'^{\varepsilon,N}$ leads to

$$\mathbb{E}\Big[\|B_t'^{\varepsilon,N}\|_1\Big] \leq \frac{C}{\sqrt{N\varepsilon^2}} \sum_{\ell=1}^d \int_{\mathbb{R}^d} \sqrt{\mathbb{E}\Big[(G_\varepsilon^\ell(x-\xi_t^1))^2\Big]}dx = \frac{C}{\sqrt{N\varepsilon^2}} \sum_{\ell=1}^d \frac{\sqrt{\int_{\mathbb{R}^d} |\frac{\partial K}{\partial x_\ell}(x)|^2 dx}}{\sqrt{\varepsilon^d}} \int_{\mathbb{R}^d} \sqrt{H_\varepsilon^\ell * p_t}(x)dx\ , \tag{5.19}$$

where $H_\varepsilon^\ell, \ell = 1, \cdots, d$ denotes the probability densities on $\mathbb{R}^d$ such that for almost all $x \in \mathbb{R}^d$, $H_\varepsilon^\ell(x) := \frac{1}{\int_{\mathbb{R}^d} |\frac{\partial K}{\partial x_\ell}(x)|^2 dx} \frac{1}{\varepsilon^d} |\frac{\partial K}{\partial x_\ell}(\frac{x}{\varepsilon})|^2$. Applying again Lemma 6.2 with $G = \frac{|\frac{\partial K}{\partial x_\ell}|^2}{\|\frac{\partial K}{\partial x_\ell}\|_2^2}, \ell = 1, \cdots, d$ and $f = p_t$ we obtain

$$\mathbb{E}\Big[\|B_t'^{\varepsilon,N}\|_1\Big] \leq \frac{C}{\sqrt{N\varepsilon^{d+2}}}\ , \quad \text{for } \varepsilon \text{ small enough.} \tag{5.20}$$

Step 2: Bounding $\mathbb{E}\|A_t^{\varepsilon,N}\|_1$ and $\mathbb{E}\|A_t'^{\varepsilon,N}\|_1$. Recall that $A_t^{\varepsilon,N}(x) = K_\varepsilon * (\gamma_t^{\varepsilon,N} - \tilde{\gamma}_t^{\varepsilon,N})(x)$ and $A_t'^{\varepsilon,N}(x) = \frac{1}{\varepsilon}\sum_{\ell=1}^d |G_\varepsilon^\ell * (\gamma_t^{\varepsilon,N} - \tilde{\gamma}_t^{\varepsilon,N})|(x)$, which yields

$$\mathbb{E}\Big[\|A_t^{\varepsilon,N}\|_1\Big] + \mathbb{E}\Big[\|A_t'^{\varepsilon,N}\|_1\Big] \leq \frac{C}{\varepsilon}\mathbb{E}\Big[\|\gamma_t^{\varepsilon,N} - \tilde{\gamma}_t^{\varepsilon,N}\|_{TV}\Big]\ . \tag{5.21}$$

We are now interested in bounding the r.h.s. of (5.21).

Recalling (5.1), (5.12) and inequality (2.3), we have

$$\begin{aligned}
\mathbb{E}\Big[\|\gamma_t^{\varepsilon,N} - \tilde{\gamma}_t^{\varepsilon,N}\|_{TV}\Big] &= \frac{1}{N}\sum_{i=1}^N \mathbb{E}\Big[|V_t(\xi^i, u^{\varepsilon,N}, \nabla u^{\varepsilon,N}) - V_t(\xi^i, u^\varepsilon, \nabla u^\varepsilon)|\Big] \\
&\leq C\mathbb{E}\left[\int_0^t \left(|u_s^{\varepsilon,N} - u_s^\varepsilon|(\xi_s^1) + |\nabla u_s^{\varepsilon,N} - \nabla u_s^\varepsilon|(\xi_s^1)\right) ds\right] \\
&\leq C\int_0^t \left(\mathbb{E}\Big[|K_\varepsilon * (\gamma_s^{\varepsilon,N} - \tilde{\gamma}_s^{\varepsilon,N})(\xi_s^1)|\Big] + \mathbb{E}\Big[|K_\varepsilon * (\tilde{\gamma}_s^{\varepsilon,N} - \gamma_s^\varepsilon)(\xi_s^1)|\Big] ds\right)\ , \\
&\quad + \frac{C}{\varepsilon}\sum_{\ell=1}^d \int_0^t \left(\mathbb{E}\Big[|G_\varepsilon^\ell * (\gamma_s^{\varepsilon,N} - \tilde{\gamma}_s^{\varepsilon,N})(\xi_s^1)|\Big] + \mathbb{E}\Big[|G_\varepsilon^\ell * (\tilde{\gamma}_s^{\varepsilon,N} - \gamma_s^\varepsilon)(\xi_s^1)|\Big]\right) ds.
\end{aligned} \tag{5.22}$$

By inequality (6.18) in Lemma 6.4, there exists a finite constant $C > 0$ such that $\|p_s\|_\infty \leq C\|u_0\|_\infty$ for all $s \in [0, T]$. Thus using inequality (5.18), we obtain

$$\begin{aligned}
&\mathbb{E}\Big[|K_\varepsilon * (\tilde{\gamma}_s^{\varepsilon,N} - \gamma_s^\varepsilon)(\xi_s^1)|\Big] \\
&\leq \frac{1}{N}\mathbb{E}\Big[\,\big|K_\varepsilon(0)V_s(\xi^1, u^\varepsilon(\xi^1), \nabla u^\varepsilon(\xi^1)) - \mathbb{E}[K_\varepsilon(\xi_s^1 - \xi_s^2)V_s(\xi^2, u^\varepsilon(\xi^2), \nabla u^\varepsilon(\xi^2))\,|\,\xi^2]\big|\,\Big] \\
&+ \frac{N-1}{N}\frac{1}{N-1}\int_{\mathbb{R}^d}\left|\sum_{i=2}^N \Big[K_\varepsilon(x - \xi_s^i)V_s(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)) - \mathbb{E}[K(x-\xi_s^i)V_s(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i))]\Big]\right| p_s(x)dx \\
&\leq \frac{C}{N\varepsilon^d} + \frac{N-1}{N}\frac{C}{\sqrt{(N-1)\varepsilon^d}} \\
&\leq \frac{C}{\sqrt{N\varepsilon^d}} \qquad (\text{for} \quad (N \text{ and } N\varepsilon^d) \quad \text{sufficiently large}),\ s \in [0, T]\ .
\end{aligned} \tag{5.23}$$

Similarly we get

$$\sum_{\ell=1}^{d} \mathbb{E}\Big[|\frac{1}{\varepsilon}G_\varepsilon^\ell * (\tilde{\gamma}_s^{\varepsilon,N} - \gamma_s^\varepsilon)(\xi_s^1)|\Big] \leq \frac{C}{\sqrt{N\varepsilon^{d+2}}}, \quad s \in [0,T] \,. \tag{5.24}$$

Moreover, for all $s \in [0,T]$, the boundedness of $|K|$ and $|\nabla K|$ implies

$$\mathbb{E}\Big[|K_\varepsilon * (\gamma_s^{\varepsilon,N} - \tilde{\gamma}_s^{\varepsilon,N})(\xi_s^1)|\Big] + \sum_{\ell=1}^{d} \mathbb{E}\Big[|\frac{1}{\varepsilon}G_\varepsilon^\ell * (\gamma_s^{\varepsilon,N} - \tilde{\gamma}_s^{\varepsilon,N})(\xi_s^1)|\Big] \leq \frac{C}{\varepsilon^{d+1}}\Big[\|\gamma_s^{\varepsilon,N} - \tilde{\gamma}_s^{\varepsilon,N}\|_{TV}\Big] \,. \tag{5.25}$$

Injecting inequalities (5.23) (5.24) and (5.25) into (5.22) gives

$$\mathbb{E}\Big[\|\gamma_t^{\varepsilon,N} - \tilde{\gamma}_t^{\varepsilon,N}\|_{TV}\Big] \leq \frac{C}{\sqrt{N\varepsilon^{d+2}}} + \frac{C}{\varepsilon^{d+1}}\int_0^t \mathbb{E}\Big[\|\gamma_s^{\varepsilon,N} - \tilde{\gamma}_s^{\varepsilon,N}\|_{TV}\Big] ds \,.$$

By Gronwall's lemma we obtain $\mathbb{E}\Big[\|\gamma_t^{\varepsilon,N} - \tilde{\gamma}_t^{\varepsilon}\|_{TV}\Big] \leq \frac{C}{\sqrt{N\varepsilon^{d+2}}} e^{\frac{C}{\varepsilon^{d+1}}}$, which together with (5.21) completes the proof by implying the inequality

$$\mathbb{E}[\|A_t^{\varepsilon,N}\|_1] + \mathbb{E}[\|A_t'^{\varepsilon,N}\|_1] \leq \frac{C}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}} \,. \tag{5.26}$$

□

Corollary 5.3. *Assume that the same assumptions as in Proposition 5.2 are fulfilled.*
If $\varepsilon \to 0$, $N \to +\infty$ such that $\frac{1}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}} \to 0$, (where C is the constant coming from Proposition 5.2) then

$$\mathbb{E}\Big[\|u_t^{\varepsilon,N} - u_t\|_1\Big] + \mathbb{E}\Big[\|\nabla u_t^{\varepsilon,N} - \nabla u_t\|_1\Big] \longrightarrow 0 \,. \tag{5.27}$$

Proof. Let us fix $\varepsilon > 0$, $N \in \mathbb{N}^\star$, $t \in [0,T]$. The proof is based on the bound

$$\begin{aligned} \mathbb{E}\Big[\|u_t^{\varepsilon,N} - u_t\|_1\Big] + \mathbb{E}\Big[\|\nabla u_t^{\varepsilon,N} - \nabla u_t\|_1\Big] &\leq \mathbb{E}\Big[\|u_t^{\varepsilon,N} - u_t^\varepsilon\|_1\Big] + \mathbb{E}\Big[\|\nabla u_t^{\varepsilon,N} - \nabla u_t^\varepsilon\|_1\Big] \\ &\quad + \|u_t^\varepsilon - u_t\|_1 + \|\nabla u_t^\varepsilon - \nabla u_t\|_1 \,, \\ &\leq \frac{C}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}} + \|u_t^\varepsilon - u_t\|_1 + \|\nabla u_t^\varepsilon - \nabla u_t\|_1 \,, \end{aligned} \tag{5.28}$$

where we have used Proposition 5.2 for the second inequality above.
Taking into account Theorem 4.5 above, it appears clearly that the convergence of $u^{\varepsilon,N}$ (resp. $\nabla u^{\varepsilon,N}$) to u (resp. ∇u) will hold as soon as $\frac{1}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}} \to 0$ when $\varepsilon \to 0$, $N \to +\infty$. This concludes the proof of the corollary. □

Remark 5.4. *In the statement of Corollary 5.3 appears the condition $\frac{1}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}} \to 0$ when $\varepsilon \to 0$, $N \to +\infty$. This requires a "trade-off" between the speed of convergence of N and ε. Setting $\Phi(\varepsilon) := \varepsilon^{-(d+4)} e^{\frac{2C}{\varepsilon^{d+1}}}$, the trade-off condition can be formulated as*

$$\frac{\Phi(\varepsilon)}{N} \to 0 \quad when \quad \varepsilon \to 0, \ N \to +\infty. \tag{5.29}$$

An example of such trade-off between N and ε can be given by the relation $\varepsilon(N) = (\frac{1}{log(N)})^{\frac{1}{d+4}}$.

6 Appendix

6.1 General inequalities

If f is a probability density on $\mathbb{R}^d$, $I(f)$ denotes the quantity $I(f) := \int_{\mathbb{R}^d} |x|^{d+1} f(x) dx$.

Lemma 6.1 (Multidimensional Carlson's inequality)**.** *Let f be a probability density on $\mathbb{R}^d$ such that $I(f) < \infty$, then*

$$\int_{\mathbb{R}^d} \sqrt{f(x)}dx \le A_d\, I(f)^{\frac{d}{2(d+1)}} \qquad \textit{where} \qquad A_d = \left(\frac{(2\pi)^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} \right)^{1/2} . \tag{6.1}$$

We refer to [14], where a more precise estimate is proved. From Carlson's inequality, we deduce the following lemma.

Lemma 6.2. *Let G and f be two probability densities defined on $\mathbb{R}^d$ such that*

$$I(G) < \infty\,, \quad \textit{and} \quad I(f) < \infty\,. \tag{6.2}$$

Then for any strictly positive real $\varepsilon \le (1/I(G))^{\frac{1}{d+1}}$,

$$\int_{\mathbb{R}^d} \sqrt{(G_\varepsilon * f)(x)}\, dx \le 2^{\frac{d}{2}}\, A_d\, [1 + I(f)] \quad \textit{where} \quad A_d = \left(\frac{(2\pi)^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} \right)^{1/2} , \tag{6.3}$$

where $G_\varepsilon(\cdot) := \frac{1}{\varepsilon^d} G(\frac{\cdot}{\varepsilon})$.

Proof. By Carlson's inequality (6.1) we have

$$\int_{\mathbb{R}^d} \sqrt{(G_\varepsilon * f)(x)}dx \le A_d\, [I(G_\varepsilon * f)]^{\frac{d}{2(d+1)}} . \tag{6.4}$$

Then, by Minkowski's inequality,

$$\begin{aligned}
[I(G_\varepsilon * f)]^{\frac{1}{d+1}} &= \Big[\int_{\mathbb{R}^d \times \mathbb{R}^d} |x|^{d+1}\, G_\varepsilon(x-y)\, f(y) dy\, dx \Big]^{\frac{1}{d+1}} \\
&= \Big[\int_{\mathbb{R}^d \times \mathbb{R}^d} |y + u\varepsilon|^{d+1}\, G(u)\, f(y) dy\, du \Big]^{\frac{1}{d+1}} \\
&\le I(f)^{\frac{1}{d+1}} + \varepsilon\, I(G)^{\frac{1}{d+1}} .
\end{aligned}$$

Since $x \in \mathbb{R}^+ \mapsto x^d$ is convex, it follows

$$I(G_\varepsilon * f)^{\frac{d}{2(d+1)}} \quad \le \quad 2^{\frac{d-1}{2}} \left[[I(f)]^{\frac{d}{d+1}} + \varepsilon^d\, [I(G)]^{\frac{d}{d+1}} \right]^{\frac{1}{2}} .$$

Hence, as soon as $\varepsilon \le (1/I(G))^{\frac{1}{d+1}}$, we have

$$[I(G_\varepsilon * f)]^{\frac{d}{2(d+1)}} \le 2^{\frac{d}{2}}\, [1 + I(f)]\,, \tag{6.5}$$

which, owing to (6.4), concludes the proof. ☐

Lemma 6.3. *Let H be a density kernel on $\mathbb{R}^d$ satisfying*

$$H \ge 0, \int_{\mathbb{R}^d} H(x)\, dx = 1\,, \quad \int_{\mathbb{R}^d} x\, H(x)\, dx = 0\,. \tag{6.6}$$

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a real-valued function. For any $\varepsilon > 0$, we consider the function H_ε given by

$$H_\varepsilon(\cdot) := \frac{1}{\varepsilon^d} H\left(\frac{\cdot}{\varepsilon}\right) . \tag{6.7}$$

If $a := \frac{1}{2}\int_{\mathbb{R}^d} |x|^2\, H(x)\, dx < \infty$ (resp. $\tilde{a} := \int_{\mathbb{R}^d} |x|\, H(x)\, dx < \infty$) and $f \in W^{2,p}$ (resp. $f \in W^{1,p}$) for some integer $p \ge 1$, then for any $\varepsilon > 0$,

$$\|H_\varepsilon * f - f\|_p \le \varepsilon^2\, a \sum_{i,j=1}^{d} \|\partial_i \partial_j f\|_p\,. \tag{6.8}$$

$$\Big(\textit{resp.} \quad \|H_\varepsilon * f - f\|_p \le \varepsilon\, \tilde{a} \sum_{i=1}^{d} \|\partial_i f\|_p\Big). \tag{6.9}$$

Proof. The proof is modeled on [14] and it is only written in the case $f \in W^{2,p}$. The case $f \in W^{1,p}$ follows exactly the same reasoning. We omit the corresponding details.
For $\varepsilon > 0$ and any integer $1 \leq i < j \leq d$ let us introduce the real-valued function $L_\varepsilon^{i,j}$ defined on $\mathbb{R}^d$ with values in $\bar{\mathbb{R}}_+$, associated with H such that for almost all $x \in \mathbb{R}^d$,

$$L_\varepsilon^{i,j}(x) = \frac{x_i x_j}{\varepsilon^2} \int_0^1 \frac{1-t}{t^2} H_{\varepsilon t}(x)\, dt, \tag{6.10}$$

where x_i is the i-th coordinate of x and H_t given by (6.7). Observe that, for any $\varepsilon > 0$, $1 \leq i < j \leq d$,

$$\|L_\varepsilon^{i,j}\|_1 = \int_{\mathbb{R}^d} |L_\varepsilon^{i,j}(x)| dx \leq a, \tag{6.11}$$

which implies that $L_\varepsilon^{i,j} < \infty$ a.e.

Developing f according to the Lagrange expansion up to order two, yields, for almost all $(x, y) \in (\mathbb{R}^d)^2$,

$$f(x-y) = f(x) - \sum_{i=1}^d (\partial_i f)(x) y_i + \sum_{i,j=1}^d \int_0^1 (1-t)\partial_i\partial_j f(x - ty) y_i y_j\, dt\ .$$

Integrating this expression against H_ε w.r.t. y and using the symmetry of H, yields for almost all $x \in \mathbb{R}^d$,

$$\begin{aligned}
(H_\varepsilon * f)(x) - f(x) &= \int_{\mathbb{R}^d} [f(x-y) - f(x)]\, H_\varepsilon(y)\, dy \\
&= \sum_{i,j=1}^d \int_{\mathbb{R}^d} \int_0^1 (1-t)\partial_i\partial_j f(x-ty) y_i y_j\, dt\, H_\varepsilon(y)\, dy \\
&= \varepsilon^2 \sum_{i,j=1}^d \int_{\mathbb{R}^d} \partial_i\partial_j f(x-u) \frac{u_i u_j}{\varepsilon^2} \int_0^1 \frac{1-t}{t^2} H_{\varepsilon t}(u)\, dt\, du \\
&= \varepsilon^2 \sum_{i,j=1}^d (L_\varepsilon^{i,j} * (\partial_i\partial_j f))(x)\ ,
\end{aligned} \tag{6.12}$$

Taking the L^p norm in equality (6.12), Young's inequality yields

$$\|H_\varepsilon * f - f\|_p \leq \varepsilon^2 \sum_{i,j=1}^d \|\partial_i\partial_j f\|_p \|L_\varepsilon^{i,j}\|_1\ ,$$

which gives the result by recalling (6.11). □

6.2 About transition kernels

In the following lemma, we state well-known technical properties about the transition probability function of a diffusion process. All the statements below are established in [10].

Lemma 6.4. *We assume here the validity of Assumption 2. Consider a stochastic process Z, solution of the SDE*

$$Z_t = Z_0 + \int_0^t \Phi(s, Z_s) dW_s + \int_0^t g(s, Z_s) ds\ , \tag{6.13}$$

with Z_0 a random variable admitting a bounded density u_0. $P(s, x_0, t, \Gamma)$ denotes its transition probability function, for all $(s, x_0, t, \Gamma) \in [0, T] \times \mathbb{R}^d \times [0, T] \times \mathcal{B}(\mathbb{R}^d)$. The following statements hold.

1. *The transition probability function P admits a density, i.e. there exists a Borel function $p : (s, x_0, t, x) \mapsto p(s, x_0, t, x)$ such that for all $(s, x_0, t) \in [0, T] \times \mathbb{R}^d \times [0, T]$,*

$$P(s, x_0, t, \Gamma) = \int_\Gamma p(s, x_0, t, x)dx \quad , \quad \Gamma \in \mathcal{B}(\mathbb{R}^d) \; . \tag{6.14}$$

2. *The function p satisfies (in the classical sense) Kolmogorov backward (B) and forward (F) equations: i.e. $(s, x_0) \mapsto p(s, x_0, x, t)$ belongs to $C^{1,2}([0, t[\times\mathbb{R}^d, \mathbb{R})$ and satisfies*

$$(B) \left\{ \begin{array}{l} \partial_s p + L_s p = 0 \; , \quad 0 \le s < t \le T \\ p(s, x_0, t, x) \underset{s\uparrow t}{\longrightarrow} \delta_x \; , \textit{weakly} \; , \end{array} \right. \tag{6.15}$$

where for $s \in [0, T]$, $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $L_s\varphi$ has been defined in (2.4);
$(t, x) \mapsto p(s, x_0, t, x)$ is in $C^{1,2}(]s, T] \times \mathbb{R}^d, \mathbb{R})$ and satisfies

$$(F) \left\{ \begin{array}{l} \partial_t p = L_t^* p \; , \quad 0 \le s < t \le T \\ p(s, x_0, t, x) \underset{t\downarrow s}{\longrightarrow} \delta_{x_0} \; , \textit{weakly} \; , \end{array} \right. \tag{6.16}$$

where for $t \in [0, T]$, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $L_t^*\varphi$ has been defined by (2.5).
In particular, p is twice continuously differentiable w.r.t. x_0 and x.

3. *There exist real constants $C_u, c_u > 0$ such that, for $0 \le s < t \le T$, $(x_0, x) \in \mathbb{R}^d \times \mathbb{R}^d$ and for all multi-index $m := (m_1, m_2)$ whose length $|m| := m_1 + m_2$ is less or equal to 2, we have*

$$\Big| \frac{\partial^{m_1}}{\partial z_i} \frac{\partial^{m_2}}{\partial x_j} p(s, z, t, x) \Big| \le \frac{C_u}{(t-s)^{\frac{d+|m|}{2}}} e^{-c_u \frac{|x-z|^2}{t-s}}, \quad 0 \le s < t \le T, (z, x) \in (\mathbb{R}^d)^2 \; . \tag{6.17}$$

In particular, there exists a constant $C > 0$ (only depending on C_u, c_u) such that for all $t \in [0, T]$, the law density p_t of Y_t satisfies

$$\|p_t\|_\infty \le C \|u_0\|_\infty \; , \tag{6.18}$$

where p_t is given by $\int_{\mathbb{R}^d} p(0, x_0, t, x) u_0(x_0) dx_0$.

Proof. Under Assumption 2, the results are a result of Theorems 4.6, 4.7, Section 4 and Theorem 5.4, Section 5 in Chapter 6 in [10]. □

6.3 Proof of technicalities of Section 3

We give in this section the proof of Lemma 3.4.

Proof of Lemma 3.4. We only prove the direct implication since the converse follows easier with similar arguments. Without restriction of generality, we can assume that $T = N\tau$ for some integer $N \in \mathbb{N}$. The aim is to prove, for all $n \in \{1, \cdots, N\}$,

$$(H_n) \left\{ \begin{array}{l} \mu(t, dx) = \int_{\mathbb{R}^d} P(0, x_0, t, dx) u_0(dx_0) + \int_0^t ds \int_{\mathbb{R}^d} P(s, x_0, t, dx) \tilde{\Lambda}(s, x_0) \mu(s, dx_0) \; , \\ \text{for all } t \in [0, n\tau] \; . \end{array} \right. \tag{6.19}$$

We are going to proceed by induction on n. For $n = 1$, formula (6.19) follows from (3.13) by taking $k = 0$. We suppose now that (H_{n-1}) holds for some integer $n \ge 1$. Then, by taking $t = (n-1)\tau$ in the first line equation of (6.19), it follows immediately that

$$\mu((n-1)\tau, dx_0) = \int_{\mathbb{R}^d} P(0, \widetilde{x_0}, (n-1)\tau, dx_0) u_0(d\widetilde{x_0}) + \int_0^{(n-1)\tau} ds \int_{\mathbb{R}^d} P(s, \widetilde{x_0}, (n-1)\tau, dx_0) \tilde{\Lambda}(s, \widetilde{x_0}) \mu(s, d\widetilde{x_0}) \ . \tag{6.20}$$

On the other hand, since (3.13) is valid for all $t \in [(n-1)\tau, n\tau]$ by plugging $k = n-1$, we obtain

$$\mu(t, dx) = \int_{\mathbb{R}^d} P((n-1)\tau, x_0, t, dx) \mu((n-1)\tau, dx_0) + \int_{(n-1)\tau}^t ds \int_{\mathbb{R}^d} P(s, x_0, t, dx) \tilde{\Lambda}(s, x_0) \mu(s, dx_0) \ , \tag{6.21}$$

for all $t \in [(n-1)\tau, n\tau]$. Inserting (6.20) in (6.21) yields

$$\begin{aligned} \mu(t, dx) &= \int_{\mathbb{R}^d} u_0(d\widetilde{x_0}) \int_{\mathbb{R}^d} P(0, \widetilde{x_0}, (n-1)\tau, dx_0) P((n-1)\tau, x_0, t, dx) \\ &\quad + \int_0^{(n-1)\tau} ds \int_{\mathbb{R}^d} \mu(s, d\widetilde{x_0}) \tilde{\Lambda}(s, \widetilde{x_0}) \int_{\mathbb{R}^d} P(s, \widetilde{x_0}, (n-1)\tau, dx_0) P((n-1)\tau, x_0, t, dx) \\ &\quad + \int_{(n-1)\tau}^t ds \int_{\mathbb{R}^d} P(s, x_0, t, dx) \tilde{\Lambda}(s, x_0) \mu(s, dx_0) \ , \ t \in [(n-1)\tau, n\tau] \ . \end{aligned} \tag{6.22}$$

Invoking the Chapman-Kolmogorov equation satisfied by the transition probability function $P(s, x_0, t, dx)$ (see e.g. expression (2.1) in Section 2.2, Chapter 2 in [25]), we have

$$P(s, \widetilde{x_0}, t, dx) = \int_{\mathbb{R}^d} P(s, \widetilde{x_0}, \theta, dz) P(\theta, z, t, dx), \quad s < \theta < t, \ (\widetilde{x_0}, z) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{6.23}$$

Applying (6.23) with $\theta = (n-1)\tau$, it follows that for all $t \in [0, n\tau]$,

$$\begin{aligned} \mu(t, dx) &= \int_{\mathbb{R}^d} u_0(d\widetilde{x_0}) P(0, \widetilde{x_0}, t, dx) \\ &\quad + \int_0^t ds \int_{\mathbb{R}^d} P(s, \widetilde{x_0}, t, dx) \tilde{\Lambda}(s, \widetilde{x_0}) \mu(s, d\widetilde{x_0}) \ . \end{aligned} \tag{6.24}$$

This shows that (H_n) holds.

□

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