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Discontinuous solutions of Hamilton-Jacobi equations on networks ^{*}

P.J. Graber[†]

C. Hermosilla[‡]

H. Zidani[§]

Abstract

This paper studies optimal control problems on networks without controllability assumptions at the junctions. The Value Function associated with the control problem is characterized as solution to a system of Hamilton-Jacobi equations with appropriate junction conditions. The novel feature of the result lies in that the controllability conditions are not needed and the characterization remains valid even when the Value Function is not continuous.

Keywords. Control problem on networks, state constraint problems, stratified structure, Hamilton-Jacobi equations equations.

1 Introduction

In this paper we are interested in a Hamilton-Jacobi approach for control problems on networks. The latter are connected closed sets constituted by 1d smooth curves with some isolated intersections that we will call junctions. This is a special case of a more general setting of control problems where the admissible trajectories are constrained to stay in a stratified domain [18]. The general motivation for control problems in networks comes from traffic flows. For this reason, it is natural to impose different dynamics and costs on each *branch* of the network. Consequently, the resulting Hamiltonian is by nature discontinuous at the junction point, which poses several difficulties in applying the known results on Hamilton-Jacobi theory.

Control problems on networks have attracted an increasing interest in the last years, and many authors have investigated the characterization of the Value Function, see for instance [1, 21, 2]. In all these papers, a common controllability assumption has been considered at the junction points. More precisely, it is assumed that around the junction points, it is always possible to move both backward and forward in each branch, as in figure 1(a). As a consequence of this assumption, the Value Function is continuous and can be characterized

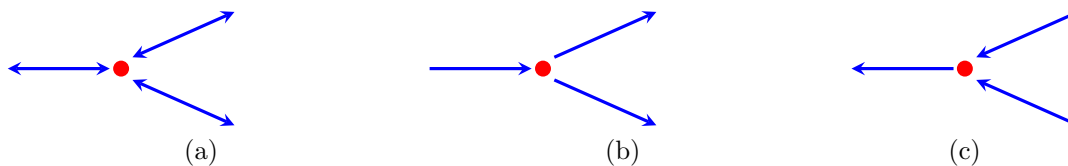


Figure 1: Different situations of transmissions conditions at a junction

by means of a system of Hamilton-Jacobi equations posed on the branches with transmission conditions at the junctions.

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[†]Department of Mathematics, Baylor University, USA. jameson_graber@baylor.edu

[‡]Department of Mathematics, LSU, USA. chermosilla@math.lsu.edu

[§]Unité des Mathématiques Appliquées (UMA), ENSTA ParisTech, France. Hasnaa.Zidani@ensta-paristech.fr

By contrast, in the present work, we consider situations where the controllability conditions are not satisfied. These include cases where the trajectories are constrained to move forward on the network without being allowed to stay on the junction and/or without having the possibility to move in both directions at the junctions (see Figures 1(b)-(c)). In addition, we generalize to multidimensional networks (called a “generalized network”) consisting of d -dimensional manifolds glued together at lower dimensional junctions (cf. [10, 20, 25]); see Definition 2.2 below.

In this setting, our main result is a characterization of the Value Function as the unique solution of a system of Hamilton-Jacobi-Bellman (HJB) equations in a *bilateral viscosity* sense (see Definition 2.3). The main difficulties here come from the fact that the constraint set (the network) has an empty interior, and the dynamics as well as the distributed cost functions are defined and continuous on each branch without being globally continuous everywhere on the network. To obtain uniqueness for the system of HJB equations, it is essential to define some *junction conditions* on the behavior of solutions where branches of the network meet (see Theorem 3.1 and Theorem 3.2). In Section 5 of the paper we show that the Value Function is the smallest bilateral viscosity solution. In Section 6 we prove that it is also the largest such solution. The main theorems are stated in Section 3.

State-constrained optimal control problems have been well studied in the literature under rather restrictive controllability constraints requiring, in particular, that the constraint sets be closures of open sets; see [31, 32, 15, 22]. A characterization of the epigraph of the Value Function is obtained without any controllability condition in [3]. Recently, a new characterization of the Value Function has been derived for control problems in stratified constraint sets with possible empty interiors [18]. To obtain this result, a weak local controllability assumption is required only on strata where a chattering phenomena may occur. The arguments apply in a quite general setting where the dynamics and the cost function are defined and Lipschitz continuous on the set of constraints. In this paper, we follow the approach of [18] to characterize the Value Function of the problem as the unique solution to a suitable Hamilton-Jacobi-Bellman equation on a generalized network. Our result includes situations where the dynamics and the cost function depend on the position on each submanifold without being Lipschitz continuous on the whole network. In these cases the arguments from [18] have to be extended in many points.

Control problems in networks of dimension 1 have been recently studied by many authors [1, 2, 21, 19]. A specific case of Eikonal equations have also been considered in [10, 11]. In all these studies the junctions are nodes (submanifolds of dimension 0) and a quite strong controllability assumption is considered at the junctions. This condition allows the admissible trajectories to move from one branch to another, and in this context it can be proved that the Value Function is continuous. The main difficulty remains the comparison principle and in particular the extension of the *variable doubling* techniques for comparing the sub- and super- viscosity solutions. Note that because of the discontinuity of the dynamics and the cost functions at the junctions, the Hamiltonian is discontinuous and the definition of the viscosity notion, as introduced by Ishii [22] can still be used, however this notion is not enough to get a uniqueness result for the corresponding HJB equation. In [1, 2, 21, 19] a junction condition is derived to describe a “transmission” condition satisfied by the Value Function at the junctions. It turns out that these transmission conditions make it possible to derive a comparison principle using a specially constructed test function at junctions, extending the classical variable doubling method. Such an approach is not amenable to the situation described in our paper, however, mainly because we dispense with the strong controllability assumption. Instead, we compare solutions directly to the Value Function.

Note that control problems in networks share some similar difficulties that one encounters when dealing with control problems in multi-domains [7, 8, 27, 26]. Indeed, in this context, the multidomains constitute a partition of the whole space \mathbb{R}^N , the dynamics and the cost functions are discontinuous and the discontinuities are located on interfaces of sub-manifolds of the same dimension. The question of transmission conditions on the interfaces is very relevant here again to get a comparison principle. However, control problem in networks possess an additional difficulty coming from the topological properties of the networks. In the context of control problems in multi-domains, the trajectories are free to move from one sub-domain to another, while in the network context, the admissible trajectories are constrained to remain in the network. Nevertheless, the method of proof used in these references inspires many of the arguments used in the present work. In

particular, the study of properties belonging to the “essential dynamics” and the “essential Hamiltonian” at the boundary between sub-domains is a common theme.

1.1 Notation and mathematical definitions

Throughout this paper, \mathbb{N} and \mathbb{R} denote the sets of Natural and Real numbers, respectively. $N \in \mathbb{N}$ is a given Natural number which remains fixed all along the exposition. We use $|\cdot|$ for the Euclidean norm and $\langle \cdot, \cdot \rangle$ for the Euclidean inner product on \mathbb{R}^N . The unit open ball $\{x \in \mathbb{R}^N : |x| < 1\}$ is indicated by \mathbb{B} and with a slight abuse of notation we write $\mathbb{B}(x, r) = x + r\mathbb{B}$. The zero vector $(0, \dots, 0)$ in \mathbb{R}^N is denoted by $\mathbf{0}$ and the empty set by \emptyset . For a set $S \subseteq \mathbb{R}^N$, $\text{int}(S)$, \overline{S} , $\text{bdry}(S)$ and $\text{co}(S)$ denote its interior, closure, boundary and convex hull, respectively. Also for S convex we denote by $r\text{-int}(S)$ and $r\text{-bdry}(S)$ its relative interior and boundary, respectively. The distance function to S is $\text{dist}_S(x) = \inf\{|x - y| \mid y \in S\}$ and in the case the infimum is attained we call the set of solution the projections of x over S and we denote it by $\text{proj}_S(x)$. The epigraph and effective domain of a given function $\varphi : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ are the sets

$$\text{epi } \varphi = \{(x, r) \in \mathbb{R}^N \times \mathbb{R} \mid r \geq \varphi(x)\} \quad \text{and} \quad \text{dom } \varphi = \{x \in \mathbb{R}^N \mid \varphi(x) \in \mathbb{R}\}.$$

Similarly, the graph and effective domain of a set-valued map $\Gamma : \mathbb{R}^N \rightrightarrows \mathbb{R}^n$ are the sets

$$\text{gr } \Gamma = \{(x, z) \in \mathbb{R}^N \times \mathbb{R}^n \mid z \in \Gamma(x)\} \quad \text{and} \quad \text{dom } \Gamma = \{x \in \mathbb{R}^N \mid \Gamma(x) \neq \emptyset\}.$$

A set-valued map $\Gamma : \mathbb{R}^N \rightrightarrows \mathbb{R}^n$ is continuous at $x \in \text{dom } \Gamma$ if it is lower semicontinuous (l.s.c.) and upper semicontinuous (u.s.c.) at x . Furthermore, it is also called locally Lipschitz continuous if for any $x \in \mathbb{R}^N$ and $\delta > 0$ there exist $L > 0$ such that

$$\forall \tilde{x}, \hat{x} \in \mathbb{B}(x, \delta), \quad \Gamma(\tilde{x}) \subseteq \Gamma(\hat{x}) + L|\tilde{x} - \hat{x}|\mathbb{B}.$$

A set $M \subseteq \mathbb{R}^N$ is a d -dimensional embedded manifold of \mathbb{R}^N with boundary if for every $x \in M$ there is an open set \mathcal{O} so that

$$M \cap \mathcal{O} = \{\tilde{x} \in \mathcal{O} \mid h_1(\tilde{x}) = \dots = h_{N-d}(\tilde{x}) = 0, h_{N-d+1}(\tilde{x}) \leq 0\},$$

where $h : \mathbb{R}^N \rightarrow \mathbb{R}^{N-d+1}$ is a smooth function whose derivative $Dh(\tilde{x})$ is surjective at any $\tilde{x} \in \mathcal{O}$. The function h is called a local defining map for M around x . In the case that the condition $h_{N-d+1}(\tilde{x}) \leq 0$ can be replaced with $h_{N-d+1}(\tilde{x}) < 0$ we simply say that M is a d -dimensional embedded manifold of \mathbb{R}^N .

For an embedded manifold of \mathbb{R}^N , if h stands for a local defining map of M around x , the tangent space to M at x , which we denote by $\mathcal{T}_M(x)$, can be identified with the set

$$\{v \in \mathbb{R}^N \mid \langle \nabla h_1(x), v \rangle = \dots = \langle \nabla h_{N-d}(x), v \rangle = 0\}.$$

For a given locally closed set $S \subseteq \mathbb{R}^N$ we write $\mathcal{T}_S^B(x)$ and $\mathcal{T}_S^C(x)$ for the Bouligand and generalized tangent cones to S at $x \in S$, which are defined via

$$\mathcal{T}_S^B(x) = \left\{ v \in \mathbb{R}^N \mid \liminf_{t \rightarrow 0^+} \frac{\text{dist}_S(x + tv)}{t} \leq 0 \right\} \quad \text{and} \quad \mathcal{T}_S^C(x) = \left\{ v \in \mathbb{R}^N \mid \limsup_{\tilde{x} \rightarrow x, t \rightarrow 0^+} \frac{\text{dist}_S(\tilde{x} + tv)}{t} \leq 0 \right\}.$$

For a l.s.c. function $\omega : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$, $\partial_V \omega(t, x)$ denotes its viscosity subdifferential, that is, the collection of $(\theta, \zeta) \in \mathbb{R} \times \mathbb{R}^N$ so that there is $\varphi \in \mathcal{C}^1((0, T) \times \mathbb{R}^N)$ such that $\partial_t \varphi(t, x) = \theta$, $\nabla_x \varphi(t, x) = \zeta$ and $\omega - \varphi$ attains a local minimum at (t, x) . If φ can be taken to be a quadratic function, we say that (θ, ζ) is a proximal subgradient of ω at (t, x) . The set of all the proximal subgradients ω at (t, x) is the proximal subdifferential, which is denoted as $\partial_P \omega(t, x)$.

2 Preliminaries

In this paper we consider optimal control problems on network-like structures. The basic model we treat is the 1-dimensional case, which is concerned with an optimization problem whose domain is a collection of piecewise smooth curves; see Figure 2. However, the approach we have adopted allows us to present the results in a general settings, where the structures are the outcome of intersecting surfaces of arbitrary dimension (see Figure 3) instead of curves.

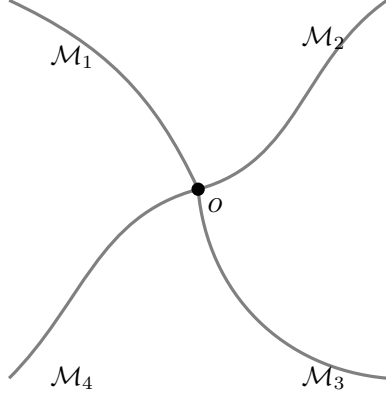


Figure 2: A 1-dimensional network in \mathbb{R}^2 having four branches $\mathcal{M}_1, \dots, \mathcal{M}_4$ and a single junction $\Upsilon = \{o\}$.

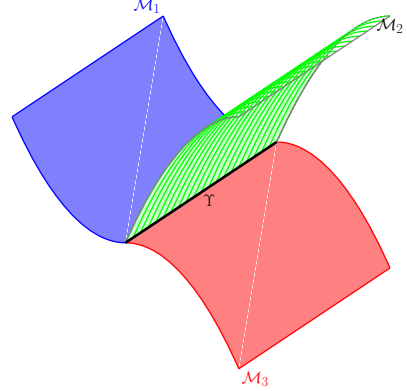


Figure 3: A 2-dimensional network in \mathbb{R}^3 having three branches $\mathcal{M}_1, \dots, \mathcal{M}_3$ and a single junction Υ .

2.1 Notions of networks

A network-like structure is a collection of smooth manifolds of two types, branches and junctions. Roughly speaking, the branches are the skeleton and the junctions are the parts that glue together the branches. In order to give a precise definition of a network-like structure, we begin by setting up the notion of junction.

Definition 2.1. Let $d \in \{1, \dots, N\}$, we say that $\Upsilon \subseteq \mathbb{R}^N$ is a d -dimensional junction if

1. Υ is a $(d - 1)$ -dimensional smooth manifold;
2. there exist $r > 0$, $p \in \mathbb{N}$ and a family $\{\mathcal{M}_1, \dots, \mathcal{M}_p\}$ of pairwise disjoint smooth manifolds such that

$$\Upsilon = (\overline{\mathcal{M}_i} \setminus \mathcal{M}_i) \cap \mathbb{B}(x, r), \quad \forall x \in \Upsilon \quad \text{and} \quad \dim(\mathcal{M}_i) = d, \quad \forall i \in \{1, \dots, p\}.$$

The collection $\{\mathcal{M}_1, \dots, \mathcal{M}_p\}$ is called the set of branches related to Υ .

Remark 2.1. In the 1-dimensional case ($d = 1$), a junction Υ contains exactly a single point (see for example Figure 2), that is, there exists $o \in \mathbb{R}^N$ so that $\Upsilon = \{o\}$, for which

$$\{o\} = (\overline{\mathcal{M}_i} \setminus \mathcal{M}_i) \cap \mathbb{B}(o, r).$$

With these concepts at hand, we now formally define a d -dimensional network as a collection of d -dimensional junctions and branches, which in addition is locally finite in space.

Definition 2.2. Let $d \in \{1, \dots, N\}$, we say that \mathcal{K} is a d -dimensional network provided there exists a locally finite pairwise disjoint collection of d -dimensional junctions $\{\Upsilon_j\}_{j \in \mathcal{J}}$ together with a pairwise disjoint family $\{\mathcal{M}_i\}_{i \in \mathcal{I}}$ of smooth manifolds satisfying the following conditions:

1. For any $x \in \mathcal{K}$ either $x \in \mathcal{M}_i$ for some $i \in \mathcal{I}$ or $x \in \Upsilon_j$ for some $j \in \mathcal{J}$.

2. For any $i \in \mathcal{I}$ there is $j \in \mathcal{J}$ so that \mathcal{M}_i is a branch related to Υ_j .
3. For any $i \in \mathcal{I}$ we have that $\overline{\mathcal{M}}_i$ is a smooth manifold with boundary.
4. For any $j \in \mathcal{J}$ there are $p_j \in \mathbb{N}$ and $i_1, \dots, i_{p_j} \in \mathcal{I}$ so that $\{\mathcal{M}_{i_1}, \dots, \mathcal{M}_{i_{p_j}}\}$ is the set of branches related to Υ_j .

Before continuing, let us make a few comments about Definition 2.2. First of all, we point out that the case where each \mathcal{M}_i is an open or half-open line segment is covered by Definition 2.2. This situation has been commonly studied in the literature in order to illustrate the main difficulties found when dealing with Hamilton-Jacobi equations on networks; see for instance [1, 2, 21, 19].

On the other hand, it is worth noticing that from Definition 2.2 we can infer that:

- the network \mathcal{K} is the pairwise disjoint union of the branches $\{\mathcal{M}_i\}_{i \in \mathcal{I}}$ and the junctions $\{\Upsilon_j\}_{j \in \mathcal{J}}$.
- $\{\mathcal{M}_i\}_{i \in \mathcal{I}}$ is a locally finite collection of d -dimensional embedded manifolds of \mathbb{R}^N .
- The boundary of each $\overline{\mathcal{M}}_i$ has finitely many connected components.
- the network \mathcal{K} is a closed set that has a countable number of connected components.
- The set $\mathcal{I} \cup \mathcal{J}$ is either finite or countably infinite.

We remark that the study of optimal control problems on d -dimensional networks is mainly motivated by traffic flow problems where it is desired to minimize a given cost-to-go functional. Nonetheless, this work has a somewhat “dual” motivation, namely the analysis of system of Hamilton-Jacobi equation.

2.2 Systems of HJB equations

Let \mathcal{I} be a countable set that indexes a collection of connected d -dimensional embedded manifolds $\{\mathcal{M}_i\}_{i \in \mathcal{I}}$ of \mathbb{R}^N . Let us consider the system of Hamilton-Jacobi equations

$$\begin{cases} -\partial_t u(t, x) + H_i(x, \nabla_x u(t, x)) = 0, & \forall (t, x) \in (0, T) \times \mathcal{M}_i, \\ u(T, x) = \psi_i(x), & \forall x \in \mathcal{M}_i, \end{cases} \quad \forall i \in \mathcal{I}, \quad (\text{HJ})$$

where each $H_i : \mathcal{M}_i \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a given Hamiltonian and each $\psi_i : \mathcal{M}_i \rightarrow \mathbb{R}$ is a given function. To avoid pathological cases, let us restrain our attention to the situation where $\{\mathcal{M}_i\}_{i \in \mathcal{I}}$ is in addition locally finite on \mathbb{R}^N and pairwise disjoint.

It is an accepted fact that classical (differentiable) solutions to the preceding system of equations rarely exist, and that the attention need to be addressed to weak notions of solutions. In our framework we are interested in l.s.c. solutions, for this reason we consider the concept of bilateral viscosity solutions.

Definition 2.3. An l.s.c. function $\omega : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ is called a bilateral viscosity solution to (HJ) provided that for each $i \in \mathcal{I}$ we have

$$-\theta + H_i(x, \zeta) = 0, \quad \forall (t, x) \in (0, T) \times \mathcal{M}_i, \quad \forall (\theta, \zeta) \in \partial_V \omega(t, x). \quad (2.1)$$

$$\liminf_{t \rightarrow T^-, \tilde{x} \rightarrow x} \omega|_{\mathcal{M}_i}(t, \tilde{x}) = \omega(T, x) = \psi_i(x), \quad \forall x \in \mathcal{M}_i. \quad (2.2)$$

$$\liminf_{t \rightarrow 0^+, \tilde{x} \rightarrow x} \omega|_{\mathcal{M}_i}(t, \tilde{x}) = \omega(0, x), \quad \forall x \in \mathcal{M}_i. \quad (2.3)$$

It is well known that if each \mathcal{M}_i is an open set ($d = N$), it is possible to construct a solution to (HJ) in the sense described above. This solution can be obtained through a suitable optimal control problem with state constraints

$$\mathcal{K} := \bigcup_{i \in \mathcal{I}} \overline{\mathcal{M}}_i. \quad (2.4)$$

As we will see shortly, this fact can be extended to the case in which d can be less than N . More precisely, let us restrict our attention to the Bellman case, which means that each Hamiltonian is determined by a set \mathcal{A}_i , a function $\mathcal{L}_i : \mathcal{M}_i \times \mathcal{A}_i \rightarrow \mathbb{R}$ and a vector field $f_i : \mathcal{M}_i \times \mathcal{A}_i \rightarrow \mathbb{R}^N$ in the following way:

$$H_i(x, \zeta) = \sup_{a \in \mathcal{A}_i} \{-\langle f_i(x, a), \zeta \rangle - \mathcal{L}_i(x, a)\}, \quad \forall i \in \mathcal{I}, \forall x \in \mathcal{M}_i, \forall \zeta \in \mathbb{R}^N. \quad (2.5)$$

In the nomenclature of optimal control theory \mathcal{A}_i is usually called the control space, \mathcal{L}_i is the running cost and f_i is the dynamics. We may also refer to each ψ_i as the final cost.

In order to construct a bilateral viscosity solution to (HJ) in the way we have mentioned earlier, let us consider a function $\psi : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ and a set-valued map $F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ that verify

$$\psi(x) = \psi_i(x) \quad \text{and} \quad F(x) = f_i(x, \mathcal{A}_i), \quad \forall i \in \mathcal{I}, \forall x \in \mathcal{M}_i. \quad (2.6)$$

We also take into account a *true Lagrangian* $\mathcal{L} : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ that satisfies

$$\mathcal{L}(x, v) = \inf_{a \in \mathcal{A}_i} \{\mathcal{L}_i(x, a) \mid f_i(x, a) = v\}, \quad \forall i \in \mathcal{I}, \forall x \in \mathcal{M}_i, \forall v \in f_i(x, \mathcal{A}_i). \quad (2.7)$$

Since we have taken the family $\{\mathcal{M}_i\}_{i \in \mathcal{I}}$ pairwise disjoint, the existence of such ψ , \mathcal{L} and F is guaranteed without ambiguity for any collection of final costs $\{\psi_i\}_{i \in \mathcal{I}}$, running-costs $\{\mathcal{L}_i\}_{i \in \mathcal{I}}$ and dynamics $\{f_i\}_{i \in \mathcal{I}}$.

Remark 2.2. *For the scope of this subsection, we do not need to prescribe F and ψ outside $\{\mathcal{M}_i\}_{i \in \mathcal{I}}$, or the values of \mathcal{L} away from $\{gr f_i(\cdot, \mathcal{A}_i)\}_{i \in \mathcal{I}}$. In other words, ψ , F and \mathcal{L} can have arbitrary values outside the aforementioned sets. However, in the sequel of the paper, we do pick a particular F , ψ and \mathcal{L} , making emphasis on the values at $\overline{\mathcal{M}_i} \setminus \mathcal{M}_i$; for more details see (2.11), (2.12) and (2.13). This has to be done in order to establish a link between a unique bilateral viscosity solution to (HJ) and a well-posed optimal control problems on a network-like structure.*

Recall that \mathcal{K} is the set determined by (2.4). Given a fixed final horizon $T > 0$ and an initial data $(t, x) \in [0, T) \times \mathcal{K}$, we denote by $\mathbb{S}_F^T(t, x)$ the collection of absolutely continuous curves $y : [t, T] \rightarrow \mathbb{R}^N$ solution of the following state constrained dynamical system:

$$\dot{y}(s) \in F(y(s)), \quad \text{for a.e. } s \in [t, T], \quad y(t) = x, \quad y(s) \in \mathcal{K}, \quad \forall s \in [t, T]. \quad (2.8)$$

We can associate to this dynamical system an optimal control problem and its respective optimal cost map. To be more precise, let us consider the Value Function $\vartheta : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ defined via

$$\vartheta(t, x) := \begin{cases} \inf \left\{ \int_t^T \mathcal{L}(y(s), \dot{y}(s)) ds + \psi(y(T)) \mid y \in \mathbb{S}_F^T(t, x) \right\} & (t, x) \in [0, T) \times \mathcal{K}, \\ \psi(x) & t = T, x \in \mathcal{K}, \\ +\infty & \text{otherwise.} \end{cases} \quad (2.9)$$

The relation between optimal control problems defined by means of open-loop controls and those written in terms of differential inclusions and *true Lagrangians* is rather well-known, we can for sake of completeness establish such relation now in our context.

Lemma 2.1. *Let $i \in \mathcal{I}$ be fixed. Suppose $\mathcal{A}_i \subseteq \mathbb{R}^{m_i}$ is compact for some $m_i \in \mathbb{N}$ and that f_i as well as \mathcal{L}_i are continuous on $\mathcal{M}_i \times \mathcal{A}_i$. Then for any $-\infty < t_1 < t_2 < +\infty$ and any absolutely continuous arc $y : [t_1, t_2] \rightarrow \mathcal{M}_i$ that verifies*

$$\dot{y}(s) \in f_i(y(s), \mathcal{A}_i), \quad \text{for a.e. } s \in [t_1, t_2],$$

there is a measurable control function $\alpha : [t_1, t_2] \rightarrow \mathcal{A}_i$ for which

$$\dot{y}(s) = f_i(y(s), \alpha(s)) \quad \text{and} \quad \mathcal{L}(y(s), \dot{y}(s)) = \mathcal{L}_i(y(s), \alpha(s)), \quad \text{for a.e. } s \in [t_1, t_2]. \quad (2.10)$$

Proof. Let us define the set-valued map $\mathbf{A} : [t_1, t_2] \rightrightarrows \mathcal{A}_i$ given by

$$\mathbf{A}(s) := \begin{cases} \{a \in \mathcal{A}_i \mid \dot{y}(s) = f_i(y(s), a)\} & \text{if } \dot{y}(s) \in f_i(y(s), \mathcal{A}_i) \\ \mathcal{A}_i & \text{otherwise,} \end{cases} \quad \forall s \in [t_1, t_2].$$

This multifunction is measurable (e.g. [5, Theorem 8.2.9]) and, since \mathcal{A}_i is compact and f_i is continuous, it has compact and nonempty images on $[t_1, t_2]$. Let us define the marginal multivalued map $\mathbf{R} : [t_1, t_2] \rightrightarrows \mathcal{A}_i$

$$\mathbf{R}(s) := \{a \in \mathbf{A}(s) \mid \mathcal{L}_i(y(s), a) = \min\{\mathcal{L}_i(y(s), \tilde{a}) \mid \tilde{a} \in \mathbf{A}(s)\}\}, \quad \forall s \in [t_1, t_2].$$

Since \mathcal{L}_i is continuous and $s \mapsto \mathbf{A}(s)$ has nonempty compact images, \mathbf{R} has compact and nonempty images on $[t_1, t_2]$ too, and in addition, it is a measurable set-valued map; see for example [5, Theorem 8.2.11]. Therefore, thanks to the Kuratowski Ryll-Nardzewski's selection theorem [4, Theorem 1.14.1], there is a measurable selection of \mathbf{R} , denoted by $\alpha : [t_1, t_2] \rightarrow \mathcal{A}_i$, which satisfies (2.10). \square

Using routine arguments in optimization and control theory it can be proved that ϑ verifies the dynamic programming principle:

$$\vartheta(t, x) = \inf \left\{ \int_t^\tau \mathcal{L}(y(s), \dot{y}(s)) ds + \vartheta(\tau, y(\tau)) \mid y \in \mathbb{S}_F^T(t, x) \right\}, \quad \forall (t, x) \in [0, T] \times \mathcal{K}, \quad \forall \tau \in [t, T].$$

The dynamic programming principle together with an adaptation of standard arguments yield to the following result of existence of bilateral viscosity solution to (HJ).

All along this paper we are going to assume the following standing assumptions:

$$\forall i \in \mathcal{I}, \mathcal{A}_i \text{ is a nonempty compact subset of } \mathbb{R}^{m_i}, \text{ with } m_i \in \mathbb{N}. \quad (H_A)$$

$$\forall i \in \mathcal{I}, \mathcal{L}_i \text{ is continuous on } \mathcal{M}_i \times \mathcal{A}_i \text{ and non negative.} \quad (H_L)$$

$$\begin{cases} i) \quad \forall i \in \mathcal{I}, f_i \text{ is continuous on } \mathcal{M}_i \times \mathcal{A}_i \text{ and } f_i(x, \mathcal{A}_i) \subseteq \mathcal{T}_{\mathcal{M}_i}(x), \quad \forall x \in \mathcal{M}_i. \\ ii) \quad \forall i \in \mathcal{I}, \forall a \in \mathcal{A}_i, \text{ the vector field } x \mapsto f_i(x, a) \text{ is locally Lipschitz continuous on } \mathcal{M}_i. \\ iii) \quad \exists c_f > 0 \text{ such that } \forall i \in \mathcal{I}, \max\{|f_i(x, a)| \mid a \in \mathcal{A}_i\} \leq c_f(1 + |x|), \quad \forall x \in \mathcal{M}_i. \end{cases} \quad (H_f)$$

Proposition 2.2. *Let $\{\mathcal{M}_i\}_{i \in \mathcal{I}}, \{\mathcal{A}_i\}_{i \in \mathcal{I}}, \{\psi_i\}_{i \in \mathcal{I}}, \{\mathcal{L}_i\}_{i \in \mathcal{I}}, \{f_i\}_{i \in \mathcal{I}}$ and \mathcal{K} be given as above and so that the standing hypotheses are satisfied, that is, (H_A) , (H_L) and (H_f) hold.*

Let $\psi : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ and $F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ verifying (2.6) as well as $\mathcal{L} : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfying (2.7). Suppose that $\mathbb{S}_F^T(t, x) \neq \emptyset$ for any $(t, x) \in [0, T] \times \mathcal{K}$. If the Value Function ϑ defined in (2.9) is l.s.c and the Hamiltonians H_i are given by (2.5), then ϑ is a bilateral viscosity solution of (HJ).

Proof. Note first that if $(t, x) \notin \text{dom } \vartheta$ then $\partial_V \vartheta(t, x) = \emptyset$ and so (2.1) holds by vacuity. Consequently, let us focus exclusively on the case $(t, x) \in \text{dom } \vartheta$. We split the rest of proof in several steps:

Step 1 Let $(t, x) \in (0, T) \times \mathcal{M}_i$ and $\varepsilon, h > 0$, then, the dynamic programming principle combined with standard arguments in viscosity theory yield to the existence of $y \in \mathbb{S}_F^T(t, x)$ such that

$$0 \geq \theta h + \int_t^{t+h} [(\dot{y}(s), \zeta) + \mathcal{L}(y(s), \dot{y}(s))] ds - \varepsilon h + o(h), \quad \forall (\theta, \zeta) \in \partial_V \vartheta(t, x).$$

Since $x \in \mathcal{M}_i$, we can find $\delta > 0$ so that $y(s) \in \mathcal{M}_i$ for any $s \in [t, t + \delta]$ and so, by Lemma 2.1, there is a measurable control $\alpha : [t, t + \delta] \rightarrow \mathcal{A}_i$ so that (2.10) holds. Furthermore, by (H_f) and (H_ψ) combined with (H_A) , for any $\zeta \in \mathbb{R}^N$ fixed, the map $\tilde{x} \mapsto H_i(\tilde{x}, \zeta)$ is uniformly continuous on any compact set that contains x . This fact implies in particular that

$$0 \geq (\theta - H_i(x, \zeta) - \varepsilon) h + o(h)|\zeta|, \quad \forall (\theta, \zeta) \in \partial_V \vartheta(t, x).$$

Thus, dividing by h in the last inequality and letting first $h \rightarrow 0^+$ and then $\varepsilon \rightarrow 0^+$ we get that the lefthand side in (2.1) is non negative, that is $\vartheta(\cdot)$ is a viscosity supersolution of (HJ) on $(0, T) \times \mathcal{M}_i$.

Step 2 Let $(t, x, a) \in (0, T) \times \mathcal{M}_i \times \mathcal{A}_i$. In the light of (H_f) , Nagumo's Theorem yields to the existence of $h > 0$ and a continuously differentiable curve $y : [t - h, t] \rightarrow \mathbb{R}^N$ satisfying:

$$\dot{y}(s) = f(y(s), a), \quad y(s) \in \mathcal{M}_i, \quad \forall s \in [t - h, t] \quad \text{and} \quad y(t) = x.$$

It is worth noticing that we can extend the curve $y(\cdot)$ to an arc of $\mathbb{S}_F^T(t - h, y(t - h))$ by just concatenating it to an element of $\mathbb{S}_F^T(t, x)$. Hence, by the dynamic programming principle we get

$$\vartheta(t - h, y(t - h)) - \vartheta(t, x) - \int_{t-h}^t \mathcal{L}(y(s), \dot{y}(s)) ds \leq 0.$$

Combining the foregoing inequality with Lemma 2.1 and standard arguments in viscosity theory, we have that the lefthand side in (HJ) is non positive, that is, the Value Function $\vartheta(\cdot)$ is a subsolution of (2.1) on $(0, T) \times \mathcal{M}_i$. Consequently, $\vartheta(\cdot)$ verifies (2.1)

Step 3 By definition, the Value Function verifies $\vartheta(T, x) = \psi_i(x)$ for any $x \in \mathcal{M}_i$ and any $i \in \mathcal{I}$. Therefore, we only need to prove that for any $x \in \mathcal{M}_i$, there is a sequence $(t_n, x_n) \in (0, T) \times \mathcal{M}_i$ such that $(t_n, x_n) \rightarrow (T, x)$ and $\vartheta(t_n, x_n) \rightarrow \vartheta(T, x)$.

Let $\{\varepsilon_n\} \subseteq (0, T)$ with $\varepsilon_n \rightarrow 0^+$. Using the same arguments as in step 2, we can show that for any $n \in \mathbb{N}$ large enough, there is $x_n \in \mathcal{M}_i$ and $y_n \in \mathbb{S}_F^T(t_n, x_n)$ with $y_n(T) = x$ and $t_n = T - \varepsilon_n$. Hence, by the definition of the Value Function, we have:

$$\vartheta(t_n, x_n) \leq \int_{t_n}^T \mathcal{L}(y_n(s), \dot{y}_n(s)) ds + \psi_i(x), \quad \forall n \in \mathbb{N} \text{ large enough.}$$

Finally, taking liminf in the last inequality and using the lower semicontinuity of ϑ we get (2.2).

Step 4 To conclude we need to prove that for any $x \in \mathcal{M}_i$, there is a sequence $(t_n, x_n) \in (0, T) \times \mathcal{M}_i$ such that $(t_n, x_n) \rightarrow (0, x)$ and $\vartheta(t_n, x_n) \rightarrow \vartheta(0, x)$.

Let $y \in \mathbb{S}_F^T(0, x)$, and by definition of the Value Function we have that

$$\vartheta(t, y(t)) \leq \int_t^T \mathcal{L}(y(s), \dot{y}(s)) ds + \psi(y(T)), \quad \forall t \in (0, T).$$

Hence, taking $\{t_n\} \subseteq (0, T)$ with $t_n \rightarrow 0^+$ and setting $x_n = y(t_n)$ we get

$$\liminf_{n \rightarrow +\infty} \vartheta(t_n, x_n) \leq \int_0^T \mathcal{L}(y(s), \dot{y}(s)) ds + \psi(y(T)).$$

Since $y \in \mathbb{S}_F^T(0, x)$ is arbitrary, taking infimum over them we get the Value Function on the right hand side, which completes the proof. □

Remark 2.3. *It is well-known that if F has convex images around \mathcal{K} and ψ is l.s.c., the Value Function is l.s.c. as well. Nonetheless, in the setting of this paper we do not impose the convexity assumption everywhere because, as we will see later, it is not a topological invariant of a network-like system. This issue can be overcome by means of the structure of the problem.*

Let us stress that solutions to (HJ) do not have to be unique because no information has been prescribed outside of $\{\mathcal{M}_i\}_{i \in \mathcal{I}}$. We have just seen that under appropriate hypotheses there exists at least one solution, which, in addition, is the Value Function of an optimal control problem. Therefore, it is of interest to study this solution in more detail and establish suitable conditions under which it is the unique bilateral viscosity solution to (HJ). To do so, we fix our attention on the case that \mathcal{K} is a d -dimensional network. In particular, we investigate junction conditions that are typical of the Value Function.

2.3 Further structural assumptions

From this point onward \mathcal{K} stands for a d -dimensional network whose junctions and branches are denoted by $\{\Upsilon_j\}_{j \in \mathcal{J}}$ and $\{\mathcal{M}_i\}_{i \in \mathcal{I}}$, respectively. It is not difficult to see that \mathcal{K} also is determined by (2.4). Indeed, by Definition 2.2 we have

$$\mathcal{K} = \bigcup_{i \in \mathcal{I}} \mathcal{M}_i \cup \bigcup_{j \in \mathcal{J}} \Upsilon_j = \bigcup_{i \in \mathcal{I}} \overline{\mathcal{M}_i}.$$

In this part we also make a particular choice for F , ψ and \mathcal{L} so that (2.6) and (2.7) are satisfied, and we disclose a set of assumptions that will ensure in particular that the hypotheses of Proposition 2.2 are satisfied. To do so, from now on we assume that for each $i \in \mathcal{I}$,

$$f_i \text{ and } \mathcal{L}_i \text{ can be continuously extended up to } \overline{\mathcal{M}_i} \times \mathcal{A}_i.$$

For sake of notation, we write such extensions in the same manner as the original maps, that is, in what follows we suppose that f_i and \mathcal{L}_i are defined on $\overline{\mathcal{M}_i} \times \mathcal{A}_i$. This means in particular that H_i given by (2.5) is also considered to be defined up to $\overline{\mathcal{M}_i} \times \mathbb{R}^N$ (keeping the same notation as well).

In our framework we allow networks to have several junctions, for this reason we introduce some special notation to indicate the branches associated with a certain junction. We then set

$$\mathcal{I}_j = \{i \in \mathcal{I} \mid \Upsilon_j \subseteq \overline{\mathcal{M}_i} \setminus \mathcal{M}_i\}, \quad \forall j \in \mathcal{J}.$$

The preference we have taken for F , ψ and \mathcal{L} is mainly motivated by the compactness of the set of trajectories as well as the lower semicontinuity of the Value Function. However, we wonder whether other choices can be made, and this certainly may lead to different junctions conditions than those we are presenting in the next section.

We begin with the choice of the dynamics F and to do so we essentially need to fix the dynamics at the junctions. We point out that dynamical systems on d -dimensional networks are allowed to have several trajectories moving in different directions along the junctions; in the 1-dimensional this does not happen because there is a unique trajectory, the constant one, that remains at the junction point. In our network models the velocities of the curves starting from a junction Υ_j are determined by the dynamics of its surrounding branches. For this reason we consider $F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ defined via

$$F(x) := \begin{cases} f_i(x, \mathcal{A}_i) & x \in \mathcal{M}_i \text{ for some } i \in \mathcal{I}, \\ \bigcup \{f_i(x, \mathcal{A}_i) \mid i \in \mathcal{I}_j\} & x \in \Upsilon_j \text{ for some } j \in \mathcal{J}, \\ \emptyset & \text{otherwise,} \end{cases} \quad \forall x \in \mathbb{R}^N. \quad (2.11)$$

In the sequel, F denotes the set-valued map defined above, and so we just write $\mathbb{S}^T(t, x)$ for the set of admissible curves of the dynamical system (2.8).

In order to ensure the semicontinuity of the Value Function it is required that the final costs $\{\psi_i\}_{i \in \mathcal{I}}$ are, in the same manner, semicontinuous and defined up to the corresponding $\overline{\mathcal{M}_i}$. Nevertheless, to ensure that the effective domain of the Value Function is the entire network we can assume as well that:

$$\text{for any } i \in \mathcal{I}, \psi_i \text{ is continuous } \overline{\mathcal{M}_i}. \quad (H_\psi)$$

Remark 2.4. *The result we present in this paper can also be written for final cost that are merely l.s.c. on the corresponding $\overline{\mathcal{M}_i}$. However, in such case, the characterization of the Value Function have to be understood in a subclass of functions having the same effective domain. In the framework treated in this paper we always have $\text{dom } \vartheta = [0, T] \times \mathcal{K}$.*

In accordance with the definitions we have given so far, we define the global final cost via

$$\psi(x) := \begin{cases} \psi_i(x) & x \in \mathcal{M}_i \text{ for some } i \in \mathcal{I}, \\ \min \{\psi_i(x) \mid i \in \mathcal{I}_j\} & x \in \Upsilon_j \text{ for some } j \in \mathcal{J}, \\ +\infty & \text{otherwise,} \end{cases} \quad \forall x \in \mathbb{R}^N. \quad (2.12)$$

Notice that we have used \min instead of \inf to define ψ . To clarify this, we point out that each \mathcal{I}_j is finite. Furthermore, this also implies that ψ is continuous on \mathcal{K} , and so lower semicontinuous on \mathbb{R}^N , provided that (H_ψ) holds. Moreover, by construction, (2.6) holds for the particular choice of F and ψ we have made.

On the other hand, similarly as done for the final cost ψ , we define the global running cost on the branches in such a way that (2.7) is trivially verified and, on the junctions, we take it as the minimal possible value determined by the surrounding branches. In other words, for any $(x, v) \in \mathbb{R}^N \times \mathbb{R}^N$

$$\mathcal{L}(x, v) := \begin{cases} \inf \{ \mathcal{L}_i(x, a) \mid a \in \mathcal{A}_i, v = f_i(x, a) \} & x \in \mathcal{M}_i \text{ for some } i \in \mathcal{I}, \\ \inf \{ \mathcal{L}_i(x, a) \mid i \in \mathcal{I}_j, a \in \mathcal{A}_i, v = f_i(x, a) \} & x \in \Upsilon_j \text{ for some } j \in \mathcal{J}, \\ +\infty & \text{otherwise.} \end{cases} \quad (2.13)$$

Remark 2.5. *Note that if (H_A) and (H_ψ) hold, then by compactness arguments, the infimums in the definition of \mathcal{L} are actually attained whenever $(x, v) \in \text{gr } F$.*

We stress that under the present framework the dynamics are likely to differ from one branch to another. Therefore, it is possible that at some junction Υ_j we have that

$$\exists x \in \Upsilon_j \quad \text{so that} \quad \bigcup \{ f_i(x, \mathcal{A}_i) \mid i \in \mathcal{I}_j \} \text{ is \textbf{not} a convex subset of } \mathbb{R}^N.$$

Properties as the preceding are important to provide the existence of solutions and to ensure the lower semicontinuity of the Value Function in absence of controllability assumptions around the junctions. Therefore, this issue yields to work with (optionally) nonconvex-valued dynamics, because by imposing the convexity of $F(x)$ at every $x \in \mathcal{K}$ we risk to exclude several situations of interest. For example, by doing so the case exhibited in Figure 4 can not be treated; notice that in this example the convex hull of $F(o)$ contains the zero vector even though $F(o)$ does not.

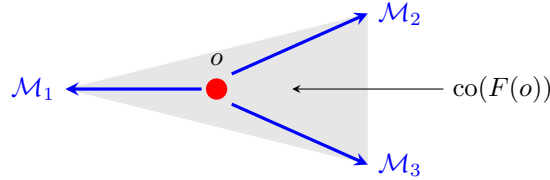


Figure 4: A case excluded if the convexity assumption is imposed.

In our setting, we are essentially facing a dynamical systems that is not well-posed in the standard theory of differential inclusions; cf. [4] or [14]. Nonetheless since the main difficulties are basically at the junctions, it is not difficult to provide some criterion for the viability of the network (existence of feasible trajectories starting from any point on the network). To do so, we mainly used the results for stratified ordinary differential equations reported in [16]. For this purpose, we can assume:

$$\forall j \in \mathcal{J}, \exists i \in \mathcal{I}_j \text{ so that } f_i(x, \mathcal{A}_i) \cap \mathcal{T}_{\overline{\mathcal{M}_i}}^C(x) \neq \emptyset, \quad \forall x \in \Upsilon_j. \quad (H_0)$$

Remark 2.6. *First of all, notice that $\mathbf{0} \in \mathcal{T}_{\overline{\mathcal{M}_i}}^C(x)$ for any $i \in \mathcal{I}$ and $x \in \overline{\mathcal{M}_i}$. This implies in particular that the following is a sufficient condition for (H_0) to hold:*

$$\forall j \in \mathcal{J}, \forall x \in \Upsilon_j, \exists i \in \mathcal{I}_j \text{ so that } \mathbf{0} \in f_i(x, \mathcal{A}_i). \quad (2.14)$$

The latter, in the 1-dimensional case, is a consequence of the usual controllability assumption at the junctions found in the literature; see for instance to [1, 2, 21, 19]. Therefore, (H_0) can be seen as a relaxation of the usual controllability hypotheses at the junctions, which allows to treat more general situations.

Furthermore, it is in some sense the minimal requirement we can ask of a network in order to well define solutions of the dynamical system. Indeed, given that each $\overline{\mathcal{M}_i}$ is a manifold with boundary, it is not difficult to see that $\mathcal{T}_{\overline{\mathcal{M}_i}}^C(x)$ agrees with $\mathcal{T}_{\overline{\mathcal{M}_i}}^B(x)$ all along $\overline{\mathcal{M}_i}$, and thus (H_0) is exactly the viability condition (cf. [4, Chapter 4]) which is a well known necessary condition for $\mathbb{S}^T(t, x)$ to be non empty at each $(t, x) \in [0, T) \times \mathcal{K}$.

To prove that the Value Function is a Real-valued l.s.c. map we still need few more structural assumptions. The statement we provide below (Proposition 2.3) is rather classical if the images of F are convex everywhere. However, as previously stated in Remark 2.3, this assumption is not made but the result still holds true. This is thanks to the structure of the problem and a convexity assumption only on the branches

$$\{(f_i(x, a), \ell) \mid a \in \mathcal{A}_i, \mathcal{L}_i(x, a) \leq \ell \leq \mathcal{L}_i^\infty(x)\} \text{ is convex } \forall x \in \mathcal{M}_i, \quad (H_1)$$

where $\mathcal{L}_i^\infty(x)$ stands for $\sup\{\mathcal{L}_i(x, a) \mid a \in \mathcal{A}_i\}$ for each $x \in \mathcal{M}_i$.

We stress that, in contrast with the 1-dimensional case, trajectories remaining at d -dimensional junctions (with $d \geq 2$) are not trivial and can move all along the junctions in different directions. For this reason extra compatibility assumptions need to be considered at the junctions. We point out that these additional conditions are generalizations of the assumptions formerly done in the literature for 1-dimensional network systems; see Remark 2.8 for more details. For this purpose, let us introduce the *tangent dynamics* to the junctions via:

$$F_j(x) = F(x) \cap \mathcal{T}_{\Upsilon_j}(x), \quad \forall j \in \mathcal{J}, \forall x \in \Upsilon_j.$$

Remark 2.7. *In 1-dimensional networks the tangent dynamics F_j are either the empty set or $\{\mathbf{0}\}$. The latter is because for each $j \in \mathcal{J}$, we have $\Upsilon_j = \{o_j\}$ for some $o_j \in \mathbb{R}^N$, which means that $\mathcal{T}_{\Upsilon_j}(o_j) = \{\mathbf{0}\}$.*

Similarly as done in [17, 18], we suppose that each F_j is regular. Furthermore, we also require that whenever $\text{dom}(F_j) \neq \emptyset$ an additional convexity condition is met. Hence, we introduce the following hypothesis:

$$\begin{cases} i) \text{ Each } F_j \text{ is a continuous multifunction on } \Upsilon_j. \\ ii) \text{ If } \text{dom}(F_j) \neq \emptyset \text{ then for any } x \in \Upsilon_j, \lambda \in [0, 1] \text{ and } v, \tilde{v} \in F(x) \text{ we have} \\ v_\lambda := \lambda v + (1 - \lambda)\tilde{v} \in \mathcal{T}_{\Upsilon_j}(x) \implies v_\lambda \in F(x) \text{ and } \lambda \mathcal{L}(x, v) + (1 - \lambda)\mathcal{L}(x, \tilde{v}) \geq \mathcal{L}(x, v_\lambda). \end{cases} \quad (H_2)$$

The foregoing hypothesis implies that either trajectories can remain for arbitrary long periods of times at the junction Υ_j or they can only pass through it. Hence, we wish to distinguish between the junctions where trajectories can slide for and where they can not. To do so, we introduce the following notation

$$\mathcal{J}_0 := \{j \in \mathcal{J} \mid \text{dom}(F_j) \neq \emptyset\}.$$

Remark 2.8. *The second point in (H₂) is a condition that ensures the existence of optimal trajectories as well as the lower semicontinuity of the Value Function. We point out that Remark 2.7 implies that in the 1-dimensional case v_λ can only be $\mathbf{0}$, and so (H₂) can be fulfilled by requiring for instance*

$$\forall j \in \mathcal{J}_0, \text{ if } \Upsilon_j = \{o_j\} \implies \mathcal{L}(o_j, v) \geq \mathcal{L}(o_j, \mathbf{0}), \quad \forall v \in F(o_j). \quad (H_2^L)$$

The preceding condition has already been considered in the literature; see for instance [1, Assumption 2.3].

In the following proposition we assume that $T > 0$ is fixed and we recall that the Value Function ϑ is given by (2.9) and determined by F , ψ and \mathcal{L} defined in (2.11), (2.12) and (2.13), respectively.

Proposition 2.3. *Let \mathcal{K} be a d -dimensional network and consider a family of control spaces $\{\mathcal{A}_i\}_{i \in \mathcal{I}}$ so that (H_A) holds. Let $\{\psi_i\}_{i \in \mathcal{I}}$, $\{\mathcal{L}_i\}_{i \in \mathcal{I}}$ and $\{f_i\}_{i \in \mathcal{I}}$ be collections of final costs, running costs and dynamics satisfying (H_ψ), (H_L) and (H_f), respectively. Assume that (H₀), (H₁) and (H₂) are also verified. Then, for every $(t, x) \in [0, T] \times \mathcal{K}$ there exists an optimal trajectory $y \in \mathbb{S}^T(t, x)$. Furthermore, the Value Function ϑ given by (2.9) is Real-valued and l.s.c. on $[0, T] \times \mathcal{K}$.*

A detailed proof of Proposition 2.3 has been postponed to Section 4.1.

3 Main results

We recall that the principal goal of this paper is to characterize the Value Function ϑ by means of Hamilton-Jacobi equations. The main difficulty is to analyze the behavior of the function at the junctions $\{\Upsilon_j\}_{j \in \mathcal{J}}$. We first study the general case of d -dimensional networks and afterwards, we present the results in the 1-dimensional networks with a single junction and branches being half-open lines. For sake of exposition all the proofs have been postponed to the last sections of paper.

3.1 d -dimensional networks

We now present the general results from which the other theorems for the 1-dimensional case can be deduced. In particular, Theorem 3.1 provides a characterization of the Value Function as the smallest bilateral viscosity solution that verifies a supersolution-type junction condition. This result is strongly related with the work of Cardaliaguet et al [12] for state constrained problem with (everywhere) convex-valued dynamics. On the other hand, Theorem 3.2 characterizes the Value Function as unique bilateral viscosity solution that satisfies some appropriate junction conditions. This result is the outcome of combining Theorem 3.1 with the work carried out in [17, 18].

3.1.1 Junction condition of supersolution-type

The junction conditions are written in terms of a smaller Hamiltonian than the usual one. This new Hamiltonian takes into account only the essential velocities, that is, the directions that are relevant for the trajectories of the control system. Formally, for any $i \in \mathcal{I}$ we define the *essential Hamiltonian* via

$$H_i^+(x, \zeta) = \sup_{a \in \mathcal{A}_i} \left\{ -\langle f_i(x, a), \zeta \rangle - \mathcal{L}_i(x, a) \mid f_i(x, a) \in \mathcal{T}_{\overline{\mathcal{M}}_i}^B(x) \right\}, \quad \forall x \in \overline{\mathcal{M}}_i, \forall \zeta \in \mathbb{R}^N. \quad (3.1)$$

Note that, since $\mathcal{T}_{\overline{\mathcal{M}}_i}^B(x)$ agrees with $\mathcal{T}_{\mathcal{M}_i}(x)$ at each $x \in \mathcal{M}_i$ and, by assumption, $f_i(x, a) \in \mathcal{T}_{\mathcal{M}_i}(x)$:

$$H_i^+(x, \zeta) = H_i(x, \zeta), \quad \forall x \in \mathcal{M}_i, \forall \zeta \in \mathbb{R}^N$$

Hence, with this definition at hand, the first result reads as follows.

Theorem 3.1. *Let \mathcal{K} be a d -dimensional network and consider a family of control spaces $\{\mathcal{A}_i\}_{i \in \mathcal{I}}$ so that (H_A) holds. Let $\{\psi_i\}_{i \in \mathcal{I}}$, $\{\mathcal{L}_i\}_{i \in \mathcal{I}}$ and $\{f_i\}_{i \in \mathcal{I}}$ be collections of final costs, running costs and dynamics satisfying (H_ψ) , $(H_{\mathcal{L}})$ and (H_f) , respectively. Assume that (H_1) , (H_0) and (H_2) are also verified. Then the Value Function of the Bolza problem on the d -dimensional network \mathcal{K} is the smallest bilateral viscosity solution to (HJ) which is $+\infty$ outside $[0, T] \times \mathcal{K}$ and that verifies the following junction conditions:*

$$\forall j \in \mathcal{J}, \forall x \in \Upsilon_j \quad : \quad \liminf_{t \rightarrow T^-, \tilde{x} \rightarrow x} \omega(t, \tilde{x}) = \omega(T, x) = \psi(x). \quad (\mathbf{C}_0)$$

$$\forall j \in \mathcal{J}, \forall x \in \Upsilon_j \quad : \quad \liminf_{t \rightarrow 0^+, \tilde{x} \rightarrow x} \omega(t, \tilde{x}) = \omega(0, x). \quad (\mathbf{C}_1)$$

$$\forall (t, x) \in (0, T) \times \Upsilon_j, \exists i \in \mathcal{I} : \quad -\theta + H_i^+(x, \zeta) \geq 0, \quad \forall (\theta, \zeta) \in \partial_V \omega(t, x). \quad (\mathbf{C}_2)$$

3.1.2 Junction condition of subsolution-type

The main goal of this section is to show additional junctions condition that are satisfied by the Value Function. These conditions will be useful to provide a complete characterization of the Value Function as unique bilateral viscosity solution to the HJB equation with appropriate junction condition.

The assumptions we consider to treat the general case are inspired by the structural conditions we encounter when dealing with one-dimensional network, and as such are trivially verified in the one-dimensional case. Recall that at any junction Υ_j , its tangent space agrees with the relative boundary of the tangent cone of any $\overline{\mathcal{M}}_i$, with $i \in \mathcal{I}_j$. Thus, the set \mathcal{A}_i can be split into three disjoint sets depending on where the velocities of the dynamics are pointing:

$$\begin{aligned} \mathbb{A}_i^+(x) &= \left\{ a \in \mathcal{A}_i \mid f_i(x, a) \in r\text{-int} \left(\mathcal{T}_{\overline{\mathcal{M}}_i}^C(x) \right) \right\} \\ \mathbb{A}_i^0(x) &= \left\{ a \in \mathcal{A}_i \mid f_i(x, a) \in \mathcal{T}_{\Upsilon_j}(x) \right\} \\ \mathbb{A}_i^-(x) &= \left\{ a \in \mathcal{A}_i \mid -f_i(x, a) \in r\text{-int} \left(\mathcal{T}_{\overline{\mathcal{M}}_i}^C(x) \right) \right\} \end{aligned}$$

Clearly we have that

$$\mathcal{A}_i = \mathbb{A}_i^-(x) \cup \mathbb{A}_i^0(x) \cup \mathbb{A}_i^+(x), \quad \forall x \in \Upsilon_j.$$

In the one-dimensional case, given that the junctions consist of a single point, the mappings defined above can be thought of as depending exclusively on the junction Υ_j . In higher dimensions and in full generality this may not be the case. For this reason, we extrapolate this remark to the case of networks of arbitrary dimensions by requiring the sets $\mathbb{A}_i^0(x)$ and $\mathbb{A}_i^-(x)$ to be independent of x , i.e. constant set-valued maps all along the junctions; note that no assumptions are required over $\mathbb{A}_i^+(x)$. In other words, we shall assume the assumption:

$$\begin{cases} i) \quad \forall j \in \mathcal{J}, \forall i \in \mathcal{I}_j, \exists \mathcal{A}_{ij}^0, \mathcal{A}_{ij}^- \subseteq \mathcal{A}_i \text{ s.t. } \mathbb{A}_i^0(x) = \mathcal{A}_{ij}^0 \text{ and } \mathbb{A}_i^-(x) = \mathcal{A}_{ij}^-, \forall x \in \Upsilon_j. \\ ii) \quad \forall x \in \Upsilon_j, \forall a \in \mathcal{A}_{ij}^0 \cup \mathcal{A}_{ij}^+, \exists r_x > 0 \text{ so that } \langle \nabla h_{N-d+1}(\tilde{x}), f_i(\tilde{x}, a) \rangle \leq 0, \forall \tilde{x} \in \mathcal{B}(x, r_x) \cap \overline{\mathcal{M}}_i. \end{cases} \quad (H_3)$$

Here $h : \mathbb{R}^{N-d+1} \rightarrow \mathbb{R}$ stands for a local defining map for $\overline{\mathcal{M}}_i$ around x . We might also assume more regularity on the running cost, that is,

$$\forall i \in \mathcal{I}, \forall a \in \mathcal{A}_i, \text{ the map } x \mapsto \mathcal{L}_i(x, a) \text{ is locally Lipschitz continuous on } \mathcal{M}_i. \quad (H_{\mathcal{L}}^+)$$

Consequently, let us consider the following Hamiltonians

$$H_i^0(x, \zeta) := \sup_{a \in \mathcal{A}_{ij}^0} \{-\langle f_i(x, a), \zeta \rangle - \mathcal{L}_i(x, a)\} \quad \text{and} \quad H_i^-(x, \zeta) := \sup_{a \in \mathcal{A}_{ij}^-} \{-\langle f_i(x, a), \zeta \rangle - \mathcal{L}_i(x, a)\}, \quad \forall x \in \Upsilon_j$$

These Hamiltonians will allow us to provide suitable junctions condition so that the Value function can be singled out as the unique bilateral viscosity solution as stated below. Note that these Hamiltonians are locally Lipschitz on the corresponding $\overline{\mathcal{M}}_i$ domain

Moreover, for technical reasons in the case $d > 1$, we also require a controllability condition on certain junctions. We recall that the reachable set of the control system, written as $\mathcal{R}(t, x; s)$, is the set of points that can be attained at time s with an admissible trajectory solution of $S^T(t, x)$.

We consider in addition $\mathcal{R}_j(t, x; \cdot)$ as the reachable set through the junction \mathcal{M}_j , that is, the set of all possible positions that can be reached with an admissible arc lying entirely on Υ_j :

$$\mathcal{R}_j(t, x; s) := \bigcup_{y \in S^T(t, x)} \{y(s) \mid y(\tau) \in \Upsilon_j, \forall \tau \in [t, s]\}, \quad \forall x \in \Upsilon_j, \forall t, s \in [0, T], t < s.$$

Therefore, the controllability hypothesis that will be required in this paper is stated as follows:

$$\begin{cases} \forall \rho > 0, \forall j \in \mathcal{J}_0, \exists \varepsilon_j, \Delta_j > 0 \text{ so that } \forall x \in \Upsilon_j, \text{ with } |x| < \rho \\ \mathcal{R}(t, x; s) \cap \Upsilon_j \subseteq \bigcup_{r \in [t, t + \Delta_j s]} \mathcal{R}_j(t, x; r), \forall t \in [0, T], \forall s \in [t, t + \varepsilon_j]. \end{cases} \quad (H_4)$$

This assumption is made in order to approximate curves that may switch between a junction and its branches infinitely many times on a short interval.

Remark 3.1. Note that (H_4) is trivial if Υ_j is a single point (since in this case, if $F_j \neq \emptyset$ then $F_j \equiv \{0\}$ and $\mathcal{R}(t, x; s) \cap \Upsilon_j = \Upsilon = \mathcal{R}_j(t, x; s)$). This implies that (H_4) is a natural extension of intrinsic properties of dynamical problems on 1-dimensional networks.

Let us also point out the fact that (H_4) can be satisfied under a simpler criterion of full *controllability condition on manifolds*. The most classical assumption of this kind of controllability is the following:

$$\forall j \in \mathcal{J}_0 \exists r_i > 0 \text{ such that } \mathcal{T}_{\Upsilon_j}(x) \cap \mathbb{B}(0, r_i) \subseteq F_j(x), \quad \forall x \in \Upsilon_j.$$

This criterion is a sufficient condition for (H_4) to be fulfilled. Indeed, this corresponds to the Petrov condition on manifolds. Hence, by adapting the classical arguments to this setting, we can see that the above-stated criterion implies the Lipschitz regularity of the minimum time function of the controlled dynamics restricted to the manifold \mathcal{M}_i , and so (H_4) follows; see for instance [6, Chapter 4.1]. However, let us emphasize that preceding criterion is only a *sufficient* condition to satisfy assumption (H_4) .

We are now in position to state the main theorem regarding the sub-solution characterization of the Value Function.

Theorem 3.2. *Let \mathcal{K} be a d -dimensional network and consider a family of control spaces $\{\mathcal{A}_i\}_{i \in \mathcal{I}}$ so that (H_A) holds. Let $\{\psi_i\}_{i \in \mathcal{I}}$, $\{\mathcal{L}_i\}_{i \in \mathcal{I}}$ and $\{f_i\}_{i \in \mathcal{I}}$ be collections of final costs, running costs and dynamics satisfying (H_ψ) , $(H_\mathcal{L})$, $(H_\mathcal{L}^+)$ and (H_f) , respectively. Assume that (H_1) , (H_0) , (H_2) (H_3) are also verified. Then the Value Function of the Bolza problem on the d -dimensional network \mathcal{K} is the unique bilateral viscosity solution to (HJ) which is $+\infty$ outside $[0, T] \times \mathcal{K}$ and that verifies (C_0) , (C_1) and (C_2) , together with the following additional junction condition*

$$\forall (t, x) \in (0, T] \times \Upsilon_j, \forall i \in \mathcal{I} : \quad -\theta + \max\{H_i^0(x, \zeta), H_i^-(x, \zeta)\} \leq 0, \quad \forall (\theta, \zeta) \in \partial_V \omega(t, x). \quad (C_3)$$

The proof of Theorem 3.2 is given in Section 6 and is divided into two parts. First, we prove that the Value Function satisfies the assumption (Lemma 6.3), and then we prove that any bilateral viscosity solution that satisfies the junctions conditions is less than or equal to the Value Function (Lemma 6.4). Hence, in the light of Theorem 3.1 the conclusion follows.

3.2 1-dimensional case

Let us now fix our attention on 1-dimensional networks ($d = 1$). Under these circumstances, and as we have already disclosed in preceding remarks, many assumptions can be substantially simplified. To the best of our knowledge, this situation is the only one that has been investigated in the current literature. We recall that Theorem 3.3 is a direct corollary of Theorem 3.1.

To illustrate our results we focus on the simplest case of branches that are half-open lines with a single junction Υ . For sake of simplicity the junction point is going to be taken as $\mathbf{0}$, the origin of the ambient space \mathbb{R}^N . Let us fix the number of branches as $p \in \mathbb{N}$, and then

$$\text{each } \mathcal{M}_i = (0, +\infty)e_i, \text{ for some } e_1, \dots, e_p \in \mathbb{R}^N \setminus \{\mathbf{0}\}. \quad (3.2)$$

In this setting, the set of indices for the branches is $\mathcal{I} = \{1, \dots, p\}$. Furthermore, we see that

$$\forall i \in \mathcal{I} : \quad \mathcal{T}_{\mathcal{M}_i}(x) = \mathbb{R}e_i, \quad \forall x \in \mathcal{M}_i \quad \text{and} \quad \mathcal{T}_{\mathcal{M}_i}^B(\mathbf{0}) = \mathcal{T}_{\mathcal{M}_i}^C(\mathbf{0}) = \overline{\mathcal{M}_i} = [0, +\infty)e_i,$$

and so, by (H_f) we can assume that for each index there exists a function $\mathcal{F}_i : [0, +\infty) \times \mathcal{A}_i \rightarrow \mathbb{R}$ so that

$$f_i(\lambda e_i, a) = \mathcal{F}_i(\lambda, a)e_i, \quad \forall i \in \{1, \dots, p\}, \quad \forall \lambda \in [0, +\infty), \quad \forall a \in \mathcal{A}_i.$$

Hence, the hypothesis (H_0) can be rephrased as

$$\exists i \in \{1, \dots, p\} \text{ so that } \mathbf{A}_i^+ := \{a \in \mathcal{A}_i \mid \mathcal{F}_i(\mathbf{0}, a) \geq 0\} \neq \emptyset. \quad (H_0^L)$$

Moreover, we have that the essential Hamiltonian at the junction can be written as

$$H_i^+(\mathbf{0}, \zeta) = \sup_{a \in \mathbf{A}_i^+} \{-\mathcal{F}_i(\mathbf{0}, a)\langle e_i, \zeta \rangle - \mathcal{L}_i(\mathbf{0}, a)\}, \quad \forall \zeta \in \mathbb{R}^N.$$

In this framework (H_0) and (H_2) can be replaced with (H_0^L) and (H_2^L) , respectively. In particular, the adaptation of Theorem 3.1 to this setting reads as follows.

Theorem 3.3. *Let \mathcal{K} be a 1-dimensional network with a single junction $\Upsilon = \{\mathbf{0}\}$ and p branches $\{\mathcal{M}_1, \dots, \mathcal{M}_p\}$ determined by (3.2). Let $\{\psi_i\}_{i \in \mathcal{I}}$, $\{\mathcal{L}_i\}_{i \in \mathcal{I}}$ and $\{f_i\}_{i \in \mathcal{I}}$ be collections of final costs, running costs and dynamics satisfying (H_ψ) , $(H_{\mathcal{L}})$, $(H_{\mathcal{L}}^+)$ and (H_f) , respectively. Assume that (H_0^L) , (H_1) and (H_2^L) are also verified. Then the Value Function of the Bolza problem on the 1-dimensional network \mathcal{K} is the smallest bilateral viscosity solution to (HJ) which is $+\infty$ outside $[0, T] \times \mathcal{K}$ verifying*

$$\liminf_{t \rightarrow T^-, x \rightarrow \mathbf{0}} \omega(t, x) = \omega(T, \mathbf{0}) = \psi(\mathbf{0}). \quad (\mathbf{C}_0^L)$$

$$\liminf_{t \rightarrow 0^+, x \rightarrow \mathbf{0}} \omega(t, x) = \omega(0, \mathbf{0}). \quad (\mathbf{C}_1^L)$$

$$\forall t \in (0, T), \exists i \in \{1, \dots, p\} : -\theta + H_i^+(\mathbf{0}, \zeta) \geq 0, \quad \forall (\theta, \zeta) \in \partial_V \omega(t, \mathbf{0}). \quad (\mathbf{C}_2^L)$$

As we have aforementioned, the controllability assumption (H_4) is an intrinsic property of 1-dimensional networks and so it is no longer a requirement. Moreover, assumption (H_3) is trivially satisfied if the following is required:

$$\mathcal{F}_i(\mathbf{0}, a) \geq 0 \implies \exists r > 0, \mathcal{F}_i(\lambda e_i, a) \geq 0, \quad \forall \lambda \in (0, r) \quad (3.3)$$

Therefore, the result that characterizes the Value Function in this context can be stated as follows.

Theorem 3.4. *Let \mathcal{K} be a 1-dimensional network with a single junction $\Upsilon = \{\mathbf{0}\}$ and p branches $\{\mathcal{M}_1, \dots, \mathcal{M}_p\}$ determined by (3.2). Let $\{\psi_i\}_{i \in \mathcal{I}}$, $\{\mathcal{L}_i\}_{i \in \mathcal{I}}$ and $\{f_i\}_{i \in \mathcal{I}}$ be collections of final costs, running costs and dynamics satisfying (H_ψ) , $(H_{\mathcal{L}})$, $(H_{\mathcal{L}}^+)$ and (H_f) , respectively. Assume that (H_0^L) , (H_1) , (H_2^L) (3.3) are also verified. Then the Value Function of the Bolza problem on the d -dimensional network \mathcal{K} is the unique bilateral viscosity solution to (HJ) which is $+\infty$ outside $[0, T] \times \mathcal{K}$ and that verifies (\mathbf{C}_0^L) , (\mathbf{C}_1^L) and (\mathbf{C}_2^L) , together with the following additional junction condition*

$$\forall (t, x) \in (0, T] \times \Upsilon_j, \forall i \in \{1, \dots, p\} : -\theta + \max\{-\mathcal{L}(\mathbf{0}, 0), H_i^-(\mathbf{0}, \zeta)\} \leq 0, \quad \forall (\theta, \zeta) \in \partial_V \omega(t, \mathbf{0}) \quad (\mathbf{C}_3^L)$$

3.2.1 Comparison with the current literature

Theorem 3.4 may be compared to other existence and uniqueness results for Hamilton-Jacobi equations on one dimensional networks or junctions [29, 1, 2, 21, 19]. These results also have generalizations to multidimensional structures [10, 20, 25], to which Theorem 3.2 may be compared. Previous works seem to be split between those which make use of strictly PDE methods in order to prove existence and uniqueness of viscosity solutions [29, 19, 20, 25] and those which make critical use of the optimal control characterization [1, 2, 21]. The present work falls into the latter category. Rather than proving a comparison principle using the ‘‘doubling of variables’’ argument and existence by Perron’s method, we instead use the tools of nonsmooth analysis and viability theory to compare solutions directly to the Value Function. In this work, we use the latter approach and analyze the characterization of solutions under fewer restrictions on the transmission conditions.

A major contribution of the present work is that we do not assume any uniform controllability assumption at the junctions. To date, the majority of works on Hamilton-Jacobi equations on networks contain some version of this hypothesis. In the one-dimensional case it can be expressed here in the form

$$[-\delta, \delta] \subset \mathcal{F}_i(\lambda, \mathcal{A}_i) \quad \forall \lambda \in [0, \infty), \quad \forall i \in \{1, \dots, p\}$$

with $\delta > 0$ independent of λ . Such a hypothesis allows trajectories to pass through junctions without getting ‘‘stuck in traffic’’—namely, it is always possible to transfer from one branch to another in a fixed finite amount of time from any point in a neighborhood of the junction. The controllability assumption, along with an assumption on continuity of the final cost, implies the continuity of the Value Function (see, e.g. [1, Proposition 2.1]). In the present work we dispense with this condition. In particular, our structural assumptions represent as closely as possible the ‘‘gluing together’’ of otherwise independent optimal control

problems on separate branches (and with possibly discontinuous final cost function). As a result, Theorems 3.1 and 3.3 represent a generalization of previous existence and uniqueness results for HJB equations on networks and junctions. The price to be paid for less restrictive transmission conditions is the continuity of the solution.

Note that in [1, 2, 21], the continuous solution is characterized as viscosity solution to (6.2) with a junction condition involving only the Hamiltonians H_i^+ . Theorem 3.2 establishes a characterization of (discontinuous) bilateral solution with junction conditions (C₂)-(C₃) involving both Hamiltonians H_i^+ and H_i^- . If the value function happens to be continuous, then the same arguments that will be developed in this paper can lead to a characterization involving a junction condition defined only with the Hamiltonian H^+ . This subject will be clarified in a separate work, as we prefer here to focus on the case of discontinuous solutions.

Our results may also be compared to recent work on transmission conditions for Hamilton-Jacobi-Bellman equations on multi-domains [7, 8, 9, 27, 26]. The main difference is that the setting in these references is a partition of Euclidean space into open sets with interfaces between them, while our setting is a collection of manifolds which generally have lower dimension than the ambient space. Moreover, in the above-mentioned literature on multi-domains problems the Value Function is also continuous, in spite of the discontinuity of the Hamiltonian at the boundary interface between sub-domains.

4 Lower semicontinuity of the Value Function

In this part we provide a proof for Proposition 2.3, which in particular encloses semicontinuity properties of the Value Function. Before going further we need to settle some structural issues related to d -dimensional networks verifying Definition 2.2. These are summarised in the following results. We recall at this point that $\mathcal{T}_S^C(x)$ is a convex cone for any $x \in S$ and it agrees with $\mathcal{T}_S^B(x)$ whenever S is a embedded manifold of \mathbb{R}^N with or without boundary.

Lemma 4.1. *Let \mathcal{K} be a d -dimensional network whose junctions and branches are given by $\{\Upsilon_j\}_{j \in \mathcal{J}}$ and $\{\mathcal{M}_i\}_{i \in \mathcal{I}}$. Then for any $j \in \mathcal{J}$ and any $i \in \mathcal{I}_j$ we have*

$$\mathcal{T}_{\Upsilon_j}(x) = \text{r-bdry} \left(\mathcal{T}_{\overline{\mathcal{M}_i}}^C(x) \right), \quad \forall x \in \Upsilon_j. \quad (4.1)$$

Furthermore, for any $x \in \Upsilon_j$ and $v \in \text{r-int} \left(\mathcal{T}_{\overline{\mathcal{M}_i}}^C(x) \right)$, there is $r > 0$ for which

$$x + (0, r] \mathbb{B}(v, r) \cap \overline{\mathcal{M}_i} \subseteq \mathcal{M}_i. \quad (4.2)$$

Proof. Let $j \in \mathcal{J}$ and $i \in \mathcal{I}_j$, and let us take $x \in \Upsilon_j$ arbitrary but fixed. Since $\overline{\mathcal{M}_i}$ is a d -dimensional manifold with boundary, we can find a \mathcal{C}^1 local defining map $h : \mathbb{R}^N \rightarrow \mathbb{R}^{N-d+1}$ for $\overline{\mathcal{M}_i}$ around x , that is, there is $r > 0$ so that $Dh(\cdot)$ has full rank on $\mathbb{B}(x, r)$ and

$$\overline{\mathcal{M}_i} \cap \mathbb{B}(x, r) = \{\tilde{x} \in \mathbb{B}(x, r) \mid h_1(\tilde{x}) = \dots = h_{N-d}(\tilde{x}) = 0, h_{N-d+1}(\tilde{x}) \leq 0\}.$$

Therefore, following [13, Corollary 10.44] we can check that

$$\mathcal{T}_{\overline{\mathcal{M}_i}}^C(x) = \{v \in \mathbb{R}^N \mid \langle \nabla h_1(x), v \rangle = \dots = \langle \nabla h_{N-d}(x), v \rangle = 0, \langle \nabla h_{N-d+1}(x), v \rangle \leq 0\}.$$

Moreover, we can also see that h is a local defining map for (the manifold without boundary) Υ_j around x , that is,

$$\Upsilon_j \cap \mathbb{B}(x, r) = \{\tilde{x} \in \mathbb{B}(x, r) \mid h_l(\tilde{x}) = 0, l = 1, \dots, N - d + 1\}.$$

Finally, (4.1) comes from [13, Corollary 10.44], because the latter yields to

$$\mathcal{T}_{\Upsilon_j}(x) = \{v \in \mathbb{R}^N \mid \langle \nabla h_l(x), v \rangle = 0, \forall l = 1, \dots, N - d + 1\}.$$

On the other hand, if there is no $r > 0$ so that (4.2) holds, we can construct two sequences $\{r_n\} \subseteq (0, 1)$ and $\{v_n\} \subseteq \mathbb{R}^N$ with $r_n \rightarrow 0$ and $v_n \rightarrow v$, so that

$$h_{N-d+1}(x + r_n v_n) = 0, \quad \forall n \in \mathbb{N}.$$

Since, $h_{N-d+1}(x) = 0$ and $h_{N-d+1}(\cdot)$ is continuously differentiable around x , we can easily check that

$$\langle \nabla h_{N-d+1}(x), v \rangle = \lim_{n \rightarrow +\infty} \frac{h_{N-d+1}(x + r_n v_n) - h_{N-d+1}(x)}{r_n} = 0,$$

which contradicts the fact that $v \in r\text{-int} \left(\mathcal{T}_{\mathcal{M}_i}^C(x) \right)$, so the conclusion follows. \square

Before continuing, we recall that \mathcal{J}_0 describes the set of junctions for which $\text{dom } F_j \neq \emptyset$.

Lemma 4.2. *Let \mathcal{K} be a d -dimensional network whose junctions and branches are given by $\{\Upsilon_j\}_{j \in \mathcal{J}}$ and $\{\mathcal{M}_i\}_{i \in \mathcal{I}}$. Suppose that $\{f_i\}_{i \in \mathcal{I}}$ satisfies (H_f) with $f_i(x, \mathcal{A}_i)$ being convex for any $i \in \mathcal{I}$ and $x \in \mathcal{M}_i$. Then, for any $j \in \mathcal{J} \setminus \mathcal{J}_0$ and any $i \in \mathcal{I}_j$ and any $x \in \Upsilon_j$ we have either*

$$f_i(x, \mathcal{A}_i) \subseteq r\text{-int} \left(\mathcal{T}_{\mathcal{M}_i}^C(x) \right), \quad \text{or} \quad f_i(x, \mathcal{A}_i) \subseteq \mathbb{R}^N \setminus \mathcal{T}_{\mathcal{M}_i}^C(x).$$

Proof. Let $i \in \mathcal{I}_j$ for which the affirmation on the statement does not hold at some $x \in \Upsilon_j$. We note first of all that $f_i(x, \mathcal{A}_i) \cap \mathcal{T}_{\Upsilon_j}(x) = \emptyset$, this is because $j \notin \mathcal{J}_0$. Hence, by Lemma 4.1, $f_i(x, \mathcal{A}_i) \cap r\text{-bdry} \left(\mathcal{T}_{\mathcal{M}_i}^C(x) \right) = \emptyset$. In the light of the preceding remark and the contradiction assumption, there are $a, \tilde{a} \in \mathcal{A}_i$ so that

$$f_i(x, a) \in r\text{-int} \left(\mathcal{T}_{\mathcal{M}_i}^C(x) \right) \quad \text{and} \quad f_i(x, \tilde{a}) \notin \mathcal{T}_{\mathcal{M}_i}^C(x).$$

Consequently, using the same notation as in the proof of Lemma 4.1,

$$\langle \nabla h_{N-d+1}(x), f_i(x, a) \rangle < 0 \quad \text{and} \quad \langle \nabla h_{N-d+1}(x), f_i(x, \tilde{a}) \rangle > 0.$$

Furthermore, by continuity of h , it is not difficult to see that $\langle \nabla h_l(x), f_i(x, \tilde{a}) \rangle = 0$, for any $l = 1, \dots, N-d$. Notice as well that we can find $\lambda \in (0, 1)$ so that

$$\langle \nabla h_l(x), \lambda f_i(x, a) + (1 - \lambda) f_i(x, \tilde{a}) \rangle = 0, \quad \forall l = 1, \dots, N-d+1.$$

Given that $f_i(x, \mathcal{A}_i)$ is convex, there is $\hat{a} \in \mathcal{A}_i$ so that $f_i(x, \hat{a}) = \lambda f_i(x, a) + (1 - \lambda) f_i(x, \tilde{a})$. Hence, we get that $f_i(x, \hat{a}) \in \mathcal{T}_{\Upsilon_j}(x)$, but this contradicts the fact that $j \notin \mathcal{J}_0$, so the proof is complete. \square

4.1 Proof of Proposition 2.3

We split the proof into four steps.

Step 1 (viability): The idea of the proof consists in selecting a stratified vector field ([16, Definition 2.4]) from the dynamical system that governs the optimal control problem at hand, and afterwards, use [16, Theorem 3.3] in order to state the existence of solutions for any $(t, x) \in [0, T) \times \mathcal{K}$.

For any $i \in \mathcal{I}$, let us select a control $a_i \in \mathcal{A}_i$ and write $g_i(\cdot)$ for the vector field $f_i(\cdot, a_i)$. Thanks to (H_f) , g_i is a continuous selection of $x \mapsto f_i(x, \mathcal{A}_i)$.

On the other hand, for any $j \in \mathcal{J}_0$ by (H_2) we have that $x \mapsto F_j(x)$ has nonempty compact convex images on Υ_j and it is in particular l.s.c. on Υ_j . Hence, by virtue of the Michael's Selection Theorem ([4, Theorem 1.11.1]) we can pick a continuous selection of $F_j(\cdot)$, that is, a continuous map $g_j : \Upsilon_j \rightarrow \mathbb{R}^N$ that verifies $g_j(x) \in F_j(x) = F(x) \cap \mathcal{T}_{\Upsilon_j}(x)$ for any $x \in \Upsilon_j$.

Note $G = \{g_j\}_{j \in \mathcal{J}_0} \cup \{g_i\}_{i \in \mathcal{I}}$ is a stratified vector field on the network \mathcal{K} for the stratification $\{\Upsilon_j\}_{j \in \mathcal{J}} \cup \{\mathcal{M}_i\}_{i \in \mathcal{I}}$. Thanks to (H_f) , this stratified vector field has linear growth. Furthermore, combining Lemma

4.1 and Lemma 4.2 with (H_0) , we check that the assumptions of [16, Theorem 3.3] are verified, and so for each $(t, x) \in [0, T) \times \mathcal{K}$ there exists an absolutely continuous curve $y : [t, T] \rightarrow \mathcal{K}$ satisfying $y(t) = x$ and

$$\dot{y}(s) = \begin{cases} g_i(y(s)) \in f_i(y(s), \mathcal{A}_i) & \text{whenever } y(s) \in \mathcal{M}_i, \\ g_j(y(s)) \in F_j(y(s)) & \text{whenever } y(s) \in \Upsilon_j, j \in \mathcal{J}_0, \end{cases} \quad \text{for a.e. } s \in [t, T].$$

So the network is a viable domain, because the sets $\{s \in [t, T] \mid y(s) \in \Upsilon_j\}$ are negligible whenever $j \notin \mathcal{J}_0$ (see Lemma 4.3).

Step 2 (effective domain of the Value Function): We claim that $\vartheta(t, x) \in \mathbb{R}$ for any $(t, x) \in [0, T) \times \mathcal{K}$. Indeed, from the previous step (viability) the Value Function is bounded from above because $\mathbb{S}^T(t, x) \neq \emptyset$, $\text{dom } \mathcal{L} = \text{gr}(F)$ and $\text{dom } \psi = \mathcal{K}$. Moreover, by (H_f) and the Gronwall's Lemma ([14, Proposition 4.1.4]) we have

$$y(s) \in \mathbb{B}(x, r(t, x)), \quad \forall s \in [t, T] \quad \text{where } r(t, x) = (1 + |x|)(e^{c_f(T-t)} - 1), \quad \forall y \in \mathbb{S}^T(t, x).$$

Besides, since we have taken each \mathcal{L}_i to be non negative in $(H_{\mathcal{L}})$, then \mathcal{L} is non negative as well and thus

$$\vartheta(t, x) \geq \inf_{i \in \mathcal{I}} \{\psi_i(\tilde{x}) \mid \tilde{x} \in \overline{\mathcal{M}}_i \cap \mathbb{B}(x, r(t, x))\}.$$

The number of branches is locally finite and each ψ_i is locally bounded from below on $\overline{\mathcal{M}}_i$, thus the righthand side is finite and so, as claimed earlier, $\vartheta(t, x) \in \mathbb{R}$.

Step 3 (existence of optimal trajectories): The preceding step implies in particular that for each $(t, x) \in [0, T) \times \mathcal{K}$ we can take a minimizing sequence $\{y_n\} \subseteq \mathbb{S}^T(t, x)$ for the problem at issue. In what follows, we assume that $(t, x) \in [0, T) \times \mathcal{K}$ as well as the minimizing sequence are given.

Gronwall's Lemma and a compactness argument ([4, Theorem 0.3.4] for instance) allow us to assert that (passing into a subsequence if necessary) we can assume that $\{y_n\}$ converges uniformly to an absolutely continuous map $y : [t, T] \rightarrow \mathcal{K}$. In addition, it can be proved that there is $\omega \in L^1([t, T]; \mathbb{R})$ so that $(\dot{y}_n, \mathcal{L}(y_n, \dot{y}_n))$ converges weakly in $L^1([t, T]; \mathbb{R}^{N+1})$ to (\dot{y}, ω) . Moreover, since $\{y_n\}$ is a minimizing sequence and ψ is l.s.c., it is not difficult to see that

$$\vartheta(t, x) \geq \int_t^T \omega(s) ds + \psi(y(T)).$$

Consequently, we only need to show that $y \in \mathbb{S}^T(t, x)$ and $\omega \geq \mathcal{L}(y, \dot{y})$ a.e. on $[t, T]$ to conclude that $y(\cdot)$ is an optimal trajectory. To do so, let us introduce the augmented dynamics

$$\Gamma(\tilde{x}) = \{(v, \ell) \in \mathbb{R}^N \times \mathbb{R} \mid v \in F(\tilde{x}), \mathcal{L}(\tilde{x}, v) \leq \ell \leq \mathcal{L}^\infty(\tilde{x})\}, \quad \forall \tilde{x} \in \mathcal{K},$$

where $\mathcal{L}^\infty(\tilde{x}) = \max\{\mathcal{L}_i(\tilde{x}, a) \mid i \in \mathcal{I}, \tilde{x} \in \overline{\mathcal{M}}_i, a \in \mathcal{A}_i\}$, for any $\tilde{x} \in \mathcal{K}$. The interest in this augmented dynamics lies in the fact that

$$(\dot{y}_n(s), \mathcal{L}(y_n(s), \dot{y}_n(s))) \in \Gamma(y_n(s)), \quad \text{for a.e. } s \in [t, T].$$

The set-valued map Γ has compact images and it is u.s.c. on \mathcal{K} , this is because F has closed graph, \mathcal{L} is l.s.c. and \mathcal{L}^∞ is u.s.c. on \mathcal{K} . Furthermore, thanks to (H_f) and $(H_{\mathcal{L}})$, for any $j \in \mathcal{J}$ and $i \in \mathcal{I}$ we also have that the restricted maps $\Gamma|_{\Upsilon_j}$ and $\Gamma|_{\mathcal{M}_i}$ are locally Lipschitz continuous on Υ_j and \mathcal{M}_i , respectively. Combining these properties, we can show that $\tilde{x} \mapsto \text{co}(\Gamma(\tilde{x}))$ is u.s.c. on \mathcal{K} having compact nonempty images.

Therefore, in the light of the Convergence Theorem ([4, Theorem 1.4.1]) we have that

$$(\dot{y}(s), \omega(s)) \in \text{co}(\Gamma(y(s))), \quad \text{a.e. on } [t, T].$$

By the convexity assumption (H_1) and by construction of F and \mathcal{L} (see (2.11) and (2.13), respectively) we have that $\text{co}(\Gamma(\tilde{x})) = \Gamma|_{\mathcal{M}_i}(\tilde{x})$ for any $i \in \mathcal{I}$ and $\tilde{x} \in \mathcal{M}_i$. Thus in particular

$$\dot{y}(s) \in f_i(y(s), \mathcal{A}_i) \text{ and } \omega(s) \geq \mathcal{L}(y(s), \dot{y}(s)) \quad \text{a.e. on } [t, T] \text{ whenever } y(s) \in \mathcal{M}_i.$$

On the other hand, with the help of the Lebesgue Differentiation Theorem it is not difficult to see that

$$\dot{y}(s) \in \mathcal{T}_{\Upsilon_j}(y(s)), \quad \text{a.e. on } [t, T] \text{ whenever } y(s) \in \Upsilon_j \text{ for some } j \in \mathcal{J}.$$

Using routine arguments we can show that (H_2) implies that $\text{co}(\Gamma(\tilde{x})) \cap \mathcal{T}_{\Upsilon_j}(\tilde{x}) \times \mathbb{R} \subseteq \Gamma(\tilde{x})$ for any $j \in \mathcal{J}_0$ and $\tilde{x} \in \Upsilon_j$. So, to conclude it only remains to show that, if $j \notin \mathcal{J}_0$, then $y(\cdot)$ can not stay at the junction Υ_j for a set of times of positive measure. Actually, as the following lemma states, this set is finite, so the proof of existence of optimal trajectories for any $(t, x) \in [0, T] \times \mathcal{K}$ finishes with the next result.

Lemma 4.3. *Let \mathcal{K} be a d -dimensional network whose junctions and branches are given by $\{\Upsilon_j\}_{j \in \mathcal{J}}$ and $\{\mathcal{M}_i\}_{i \in \mathcal{I}}$. Assume that $\{f_i\}_{i \in \mathcal{I}}$ satisfies (H_f) with $f_i(x, \mathcal{A}_i)$ being convex for any $i \in \mathcal{I}$ and $x \in \overline{\mathcal{M}_i}$. Suppose that $\{y_n\} \subseteq \mathbb{S}^T(t, x)$ is a sequence of trajectories that converges uniformly on $[t, T]$ to a continuous function $y : [t, T] \rightarrow \mathcal{K}$. Then the set $\mathbb{T}_j := \{s \in [t, T] \mid y(s) = \Upsilon_j\}$ is finite provided that $j \notin \mathcal{J}_0$.*

Proof. We argue by contradiction. Suppose first that \mathbb{T}_j is infinite with empty interior, that is, it contains no open intervals. Then $[t, T] \setminus \mathbb{T}_j$ is a countably infinite union of disjoint open intervals. Hence, for any $\varepsilon > 0$ we can find $\tilde{s}, \hat{s} \in \mathbb{T}_j$ such that

$$0 < \hat{s} - \tilde{s} < \varepsilon \quad \text{and} \quad y(s) \notin \Upsilon_j, \quad \forall s \in (\tilde{s}, \hat{s})$$

Let us take $\varepsilon > 0$ fixed but arbitrary, and $\tilde{s}, \hat{s} \in \mathbb{T}_j$ as above. By the network structure, there is $i \in \mathcal{I}_j$ so that $y(s) \in \mathcal{M}_i$ for each $s \in (\tilde{s}, \hat{s})$. Since y_n converges uniformly to y , for any $n \in \mathbb{N}$ large enough we can find $\tilde{s}_n, \hat{s}_n \in [t, T]$ so that

$$y_n(s) \in \mathcal{M}_i, \quad \forall s \in (\tilde{s}_n, \hat{s}_n) \quad \text{and} \quad (\tilde{s}_n, \hat{s}_n) \rightarrow (\tilde{s}, \hat{s})$$

In particular, $\dot{y}_n(s) \in f_i(y_n(s), \mathcal{A}_i)$ for a.e. $s \in (\tilde{s}_n, \hat{s}_n)$. Let us pick up the notation used to prove Lemma 4.1, and so let $h : \mathbb{R}^N \rightarrow \mathbb{R}^{N-d+1}$ be a local defining map for \mathcal{M}_i on $\mathbb{B}(y(\tilde{s}), r)$. Suppose that $\varepsilon > 0$ is small enough so that $y(s) \in \mathbb{B}(y(\tilde{s}), \frac{r}{4})$ for any $s \in [\tilde{s}, \hat{s}]$. Hence, for $n \in \mathbb{N}$ large enough, we can assume that $y_n(s) \in \mathbb{B}(y(\tilde{s}), \frac{r}{2})$ for any $s \in [\tilde{s}_n, \hat{s}_n]$, which means in particular that

$$h_{N-d+1}(y_n(\hat{s}_n)) - h_{N-d+1}(y_n(\tilde{s}_n)) = \int_{\tilde{s}_n}^{\hat{s}_n} \langle \nabla h_{N-d+1}(y_n(s)), \dot{y}_n(s) \rangle ds. \quad (4.3)$$

Since, $y_n(\tilde{s}_n) \rightarrow y(\tilde{s})$, $y_n(\hat{s}_n) \rightarrow y(\hat{s})$ and $y(\tilde{s}), y(\hat{s}) \in \Upsilon_j$ the lefthand side in (4.3) converges to zero. However, recall that Lemma 4.2 implies that we must either have

$$f_i(\tilde{x}, \mathcal{A}_i) \subseteq r\text{-int}(\mathcal{T}_{\overline{\mathcal{M}_i}}^C(\tilde{x})), \quad \forall \tilde{x} \in \Upsilon_j \quad \text{or else} \quad -f_i(\tilde{x}, \mathcal{A}_i) \subseteq r\text{-int}(\mathcal{T}_{\overline{\mathcal{M}_i}}^C(\tilde{x})), \quad \forall \tilde{x} \in \Upsilon_j.$$

Then reducing $r > 0$ if necessary, the latter yields to, either

$$\max_{\tilde{x} \in \mathbb{B}(y(\tilde{s}), \frac{r}{2})} \max_{a \in \mathcal{A}_i} \langle \nabla h_{N-d+1}(\tilde{x}), f_i(\tilde{x}, a) \rangle < 0 \quad \text{or else} \quad \min_{\tilde{x} \in \mathbb{B}(y(\tilde{s}), \frac{r}{2})} \min_{a \in \mathcal{A}_i} \langle \nabla h_{N-d+1}(\tilde{x}), f_i(\tilde{x}, a) \rangle > 0. \quad (4.4)$$

However, (4.4) implies that the righthand side in (4.3) can not be zero nor approximate to this value. Hence, this contradiction means that \mathbb{T}_j can not be infinite and have empty interior.

Before going further, we claim that, applying the same procedure to any y_n instead of y , it is possible to prove that $\mathbb{T}_j^n := \{s \in [t, T] \mid y_n(s) = \Upsilon_j\}$ is finite. Indeed, by the arguments exposed above, we have that if \mathbb{T}_j^n is infinite, it must have nonempty interior. However, in the light of the Lebesgue Differentiation Theorem, there is $s \in \mathbb{T}_j^n$ for which

$$\lim_{\delta \rightarrow 0} \frac{y_n(s + \delta) - y_n(s)}{\delta} = \dot{y}_n(s) \in F(y_n(s)).$$

Standard arguments allow us to show that $\dot{y}_n(s) \in \mathcal{T}_{\Upsilon_j}(y_n(s))$, which means that $y_n(s) \in F_j(y_n(s))$, getting then to a contradiction with the fact that $j \in \mathcal{J} \setminus \mathcal{J}_0$. So, \mathbb{T}_j^n is finite. In fact, we can state further that

there exists some $\varepsilon > 0$, which does not depend on n , such that if $s_1, s_2 \in \mathbb{T}_j^n$ then $|s_1 - s_2| > \varepsilon$. This can be deduced by slightly modifying the argument used in the previous paragraph; we omit the details.

Finally, let us assume that \mathbb{T}_j has nonempty interior. Hence, using the same notation as before, there are $\tilde{s}, \hat{s} \in \mathbb{T}_j$ with $\tilde{s} < \hat{s}$ so that $(\tilde{s}, \hat{s}) \subseteq \mathbb{T}_j$. Let $h : \mathbb{R}^N \rightarrow \mathbb{R}^{N-d+1}$ and $r > 0$ be as above. By reducing to a subinterval, we suppose that $y(s) \in \mathbb{B}(y(\tilde{s}), \frac{r}{4})$ for any $s \in [\tilde{s}, \hat{s}]$. By the uniform convergence hypothesis, there is $n_0 \in \mathbb{N}$ so that $y_n(s) \in \mathbb{B}(y(\tilde{s}), \frac{r}{2})$ for any $s \in [\tilde{s}, \hat{s}]$ and any $n \geq n_0$. Since \mathbb{T}_j^n is finite for any $n \in \mathbb{N}$ and its elements are uniformly spread apart, we can assume that $[\tilde{s}, \hat{s}] \cap \mathbb{T}_j^n = \emptyset$ for any $n \geq n_0$, and so, for any $n \geq n_0$ there is $i \in \mathcal{I}_j$ for which $y_n(s) \in \mathcal{M}_i$ for all $s \in [\tilde{s}, \hat{s}]$. Consequently,

$$h_{N-d+1}(y_n(\hat{s})) - h_{N-d+1}(y_n(\tilde{s})) = \int_{\tilde{s}}^{\hat{s}} \langle \nabla h_{N-d+1}(y_n(s)), \dot{y}_n(s) \rangle ds. \quad (4.5)$$

Therefore, setting

$$\beta_{\min} := \min_{\tilde{x} \in \mathbb{B}(y(\tilde{s}), \frac{r}{2})} \min_{a \in \mathcal{A}_i} \langle \nabla h_{N-d+1}(\tilde{x}), f_i(\tilde{x}, a) \rangle \text{ and } \beta_{\max} := \max_{\tilde{x} \in \mathbb{B}(y(\tilde{s}), \frac{r}{2})} \max_{a \in \mathcal{A}_i} \langle \nabla h_{N-d+1}(\tilde{x}), f_i(\tilde{x}, a) \rangle$$

we obtain

$$(\hat{s} - \tilde{s})\beta_{\min} \leq h_{N-d+1}(y_n(\hat{s})) - h_{N-d+1}(y_n(\tilde{s})) \leq (\hat{s} - \tilde{s})\beta_{\max}, \quad \forall n \geq n_0.$$

Since, $\tilde{s}, \hat{s} \in \mathbb{T}_j$, $y_n(\tilde{s}) \rightarrow y(\tilde{s})$ and $y_n(\hat{s}) \rightarrow y(\hat{s})$ we have that $h_{N-d+1}(y_n(\hat{s})) - h_{N-d+1}(y_n(\tilde{s})) \rightarrow 0$ as $n \rightarrow +\infty$. Nevertheless, by (4.4) we have that either $\beta_{\min} > 0$ or $\beta_{\max} < 0$. So a contradiction follows and the proof of the lemma is complete. \square

Step 4 (lower semicontinuity): Let $(t, x) \in [0, T] \times \mathcal{K}$ and take a sequence $\{(t_n, x_n)\} \subseteq [0, T] \times \mathcal{K}$ converging to (t, x) . By the preceding step, we can pick for each $n \in \mathbb{N}$ a trajectory $y_n \in \mathbb{S}^T(t_n, x_n)$ which realizes the optimal value at (t_n, x_n) , that is,

$$\vartheta(t_n, x_n) = \int_{t_n}^T \mathcal{L}(y_n(s), \dot{y}_n(s)) ds + \psi(y_n(T)), \quad \forall n \in \mathbb{N}. \quad (4.6)$$

We first state some useful bounds. By Gronwall's Lemma and (H_f) we have that

$$|y_n(s) - x_n| \leq (1 + |x_n|)(e^{c_f(T-t_n)} - 1), \quad \forall n \in \mathbb{N}, \forall s \in [t_n, T] \quad (4.7)$$

Therefore, since $(t_n, x_n) \rightarrow (t, x)$, we can find $R > 0$ large enough so that $y_n(s) \in \mathbb{B}(x, R)$ for any $n \in \mathbb{N}$ and $s \in [t_n, T]$. Which means that there is $L > 0$ so that

$$0 \leq |\dot{y}_n(s)| + \mathcal{L}(y_n(s), \dot{y}_n(s)) \leq L, \quad \forall n \in \mathbb{N}, \text{ for a.e. } s \in [t_n, T].$$

Let us study the simpler case $t = T$. Indeed, in these circumstances (4.7) implies that $y_n(T) \rightarrow x$, and since \mathcal{L} is uniformly bounded along the trajectories $s \mapsto y_n(s)$, taking the limit inferior on the righthand side of (4.6) and using the lower semicontinuity of ψ we get

$$\liminf_{n \rightarrow +\infty} \vartheta(t_n, x_n) \geq \psi(x).$$

The conclusion follows by recalling that $\vartheta(T, \tilde{x}) = \psi(\tilde{x})$ for any $\tilde{x} \in \mathcal{K}$.

We now focus on the case $t < T$. For any $n \in \mathbb{N}$ we write $\beta_n = \frac{T-t_n}{T-t}$ and $\gamma_n = T(\frac{t_n-t}{T-t})$, and we define

$$\tilde{y}_n(s) := y_n(\beta_n s + \gamma_n), \quad \forall s \in [t, T] \quad \text{and} \quad \omega_n(s) := \frac{1}{\beta_n} \mathcal{L}\left(\tilde{y}_n(s), \frac{1}{\beta_n} \dot{\tilde{y}}_n(s)\right), \quad \text{for a.e. } s \in [t, T].$$

Notice that the utility of these definitions lies on the fact that

$$\dot{\tilde{y}}_n \in \beta_n F(\tilde{y}_n), \text{ a.e. on } [t, T] \quad \tilde{y}_n(t) := x_n, \quad \tilde{y}_n(T) = y_n(T) \quad \text{and} \quad \int_t^T \omega_n(s) ds = \int_{t_n}^T \mathcal{L}(y_n(s), \dot{y}_n(s)) ds.$$

On the other hand, we can readily check that

$$|\dot{\tilde{y}}_n(s) - \dot{y}_n(\beta_n s + \gamma_n)| \leq \beta_n L \quad \text{and} \quad |\omega_n(s) - \mathcal{L}(y_n(\beta_n s + \gamma_n), \dot{y}_n(\beta_n s + \gamma_n))| \leq \frac{1}{\beta_n} L. \quad (4.8)$$

Therefore similarly as done in the step 3, by standard compactness arguments ([4, Theorem 0.3.4] for instance) and passing into a subsequence if necessary, we have that $\{\tilde{y}_n\}$ converges uniformly to an absolutely continuous map $y : [t, T] \rightarrow \mathcal{K}$. In addition, it can be proved that there is $\omega \in L^1([t, T]; \mathbb{R})$ so that (\tilde{y}_n, ω_n) converges weakly in $L^1([t, T]; \mathbb{R}^{N+1})$ to (y, ω) . The rest of the proof consists in showing that $(y, \omega) \in \Gamma(y)$ a.e. on $[t, T]$. To do this we use again the Convergence Theorem ([4, Theorem 1.4.1]) and (4.8) to prove first the $(y, \omega) \in \text{co}(\Gamma(y))$ a.e. on $[t, T]$ and we later use the same arguments as above in step 3 to conclude. It is worth noticing that we need to use the next modified version of Lemma 4.3.

Lemma 4.4. *Let \mathcal{K} be a d -dimensional network whose junctions and branches are given by $\{\Upsilon_j\}_{j \in \mathcal{J}}$ and $\{\mathcal{M}_i\}_{i \in \mathcal{I}}$. Suppose that $\{\tilde{y}_n\}$ is a sequence of absolutely continuous function defined on $[t, T]$ with values on \mathcal{K} and so that it converges uniformly on $[t, T]$ to a continuous function $y : [t, T] \rightarrow \mathcal{K}$. Assume that $\{f_i\}_{i \in \mathcal{I}}$ satisfies (H_f) with $f_i(x, \mathcal{A}_i)$ being convex for any $i \in \mathcal{I}$ and $x \in \overline{\mathcal{M}}_i$ and that for some sequence $\{\beta_n\} \subseteq (0, +\infty)$ converging to 1 we have*

$$\dot{\tilde{y}}_n(s) \in \beta_n F(\tilde{y}_n(s)), \quad \text{for a.e. } s \in [t, T].$$

Then the set $\mathbb{T}_j := \{s \in [t, T] \mid y(s) = \Upsilon_j\}$ is finite provided that $j \notin \mathcal{J}_0$.

The details of the preceding result are left to the reader (its proof is a routine modification of the one of Lemma 4.3) and so the proof of Proposition 2.3 is assumed to be complete.

5 Minimality of the Value Function as supersolution

Now we focus on the proof of Theorem 3.1. In order to facilitate the reading, we divide the proof into two lemmas. We first show that the Value Function verifies the junction conditions (\mathbf{C}_0) and (\mathbf{C}_2) (see Lemma 5.1), and afterwards we show that these conditions are enough to identify the Value Function as the minimal supersolution of the Hamilton-Jacobi equation (see Lemma 5.2).

Before going further, let us introduce some extra notation for understanding the proof of the next lemmas. We consider for each $i \in \mathcal{I}$ the extended dynamics $\Gamma_i : \overline{\mathcal{M}}_i \rightrightarrows \mathbb{R}^N$ defined via

$$\Gamma_i(x) := \{(f_i(x, a), \ell) \mid a \in \mathcal{A}_i, \mathcal{L}_i(x, a) \leq \ell \leq \mathcal{L}_i^\infty(x)\}, \quad \forall x \in \overline{\mathcal{M}}_i.$$

Remark 5.1. *It is not difficult to see that (H_A) , (H_f) and $(H_\mathcal{L})$ imply that Γ_i is u.s.c. with non empty compact images on $\overline{\mathcal{M}}_i$. Furthermore, by (H_1) , it also has convex images all along $\overline{\mathcal{M}}_i$.*

Similarly as for the dynamics F , we might also consider an extended dynamics $\Gamma : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ defined on the junction in the following way

$$\Gamma(x) := \begin{cases} \Gamma_i(x) & x \in \mathcal{M}_i \text{ for some } i \in \mathcal{I}, \\ \bigcup \{\Gamma_i(x) \mid i \in \mathcal{I}_j\} & x \in \Upsilon_j \text{ for some } j \in \mathcal{J}, \\ \emptyset & \text{otherwise,} \end{cases} \quad \forall x \in \mathbb{R}^N.$$

We can also see that Γ as multifunction is u.s.c. with compact (not necessarily convex) images.

Lemma 5.1. *Under the assumptions of Theorem 3.1, the Value Function of the Bolza problem on the d -dimensional network \mathcal{K} verifies (\mathbf{C}_0) and (\mathbf{C}_2) .*

Proof. To show that the Value Function satisfies (\mathbf{C}_0) in the case when $j \in \mathcal{J}_0$, we can follow the same argument used to show that the Value Function verifies (2.2) (Step 3 in the proof of Proposition 2.2), so we

skip this part of proof and focus only on the case $j \notin \mathcal{J}_0$. Under these circumstances, it may be possible that no backward trajectories start from $x \in \Upsilon_j$ at any time $t \in (0, T]$. However, in this case (\mathbf{C}_0) is a consequence of (H_ψ) . Indeed, let us fix $j \notin \mathcal{J}_0$ and $x \in \Upsilon_j$. Since there are only finitely many branches reaching Υ_j , and since, by (H_ψ) , each ψ_i is continuous up to $\overline{\mathcal{M}}_i$, we conclude that there exists $i \in \mathcal{I}_j$ so that $\psi_i(x) = \psi(x)$. Take any sequence $(t_n, x_n) \in (0, T) \times \mathcal{M}_i$ such that $(t_n, x_n) \rightarrow (T, x)$ and consider $y_n \in \mathbb{S}_F^T(t_n, x_n)$. Then, by the definition of the Value Function, we have:

$$\vartheta(t_n, x_n) \leq \int_{t_n}^T \mathcal{L}(y_n(s), \dot{y}_n(s)) ds + \psi_i(y_n(T)), \quad \forall n \in \mathbb{N}.$$

In the light of the Gronwall's Lemma and (H_f) we have that the graph of $\{y_n\}$ are uniformly bounded and $y_n(T) \rightarrow x$ as $n \rightarrow +\infty$. Moreover, the integral term in the inequality stated above vanishes as $n \rightarrow +\infty$; this is because the integrand $\mathcal{L}(y_n, \dot{y}_n(s))$ can be uniformly bounded for a.e. $s \in [t_n, T]$. Finally, taking \liminf in the inequality and using the lower semicontinuity of ϑ and the continuity of ψ_i we get (\mathbf{C}_0) .

We now turn our attention into showing that the Value Function verifies (\mathbf{C}_2) . Let $j \in \mathcal{J}$ and take $(t, x) \in (0, T) \times \Upsilon_j$ fixed but arbitrary. Thanks to Proposition 2.3 and the Dynamic Programming Principle there is $y \in \mathbb{S}^T(t, x)$ so that

$$\vartheta(t+h, y(t+h)) + \int_t^{t+h} \mathcal{L}(y(s), \dot{y}(s)) ds = \vartheta(t, x), \quad \forall h \in [0, T-t]. \quad (5.1)$$

Let us consider the absolutely continuous function $z : [t, T] \rightarrow \mathbb{R}$ defined via

$$z(t+h) := \int_t^{t+h} \mathcal{L}(y(s), \dot{y}(s)) ds, \quad \forall h \in [0, T-t].$$

Now choose any non increasing sequence $\{h_n\} \subseteq (0, T-t)$ converging to 0 so that

$$v_n := \frac{y(t+h_n) - x}{h_n} \rightarrow v \quad \text{and} \quad \ell_n := \frac{z(t+h_n)}{h_n} \rightarrow \ell, \quad \text{as } n \rightarrow +\infty.$$

The latter is always possible because F and \mathcal{L} are locally bounded thanks to (H_f) and $(H_{\mathcal{L}})$.

Let us assume for a moment that $(v, \ell) \in \Gamma_i(x)$ with $v \in \mathcal{T}_{\mathcal{M}_i}^B(x)$, for some $i \in \mathcal{I}_j$. In the light of [14, Proposition 3.4.10 and Proposition 3.4.12], for any $(\theta, \zeta) \in \partial_V \vartheta(t, x)$ and any sequence $\{(t_n, x_n)\}$ converging to (t, x) the following holds true:

$$\liminf_{n \rightarrow +\infty} \frac{\vartheta(t_n, x_n) - \vartheta(t, x) - \theta(t_n - t) - \langle \zeta, x_n - x \rangle}{|x_n - x| + |t_n - t|} \geq 0. \quad (5.2)$$

Now, setting $t_n = t + h_n$ and $x_n = y(t + h_n)$, and using (5.1) we get

$$\frac{\vartheta(t_n, x_n) - \vartheta(t, x) - \theta(t_n - t) - \langle \zeta, x_n - x \rangle}{|x_n - x| + |t_n - t|} = \frac{-z(t_n) - \theta h_n - \langle \zeta, x_n - x \rangle}{|x_n - x| + h_n}, \quad \forall n \in \mathbb{N}. \quad (5.3)$$

Furthermore, it is not difficult to see that the righthand side of the preceding equality converges to $\frac{-\ell - \theta - \langle \zeta, v \rangle}{|v|+1}$. Therefore, since (θ, ζ) is arbitrary, by virtue of (5.2), letting $n \rightarrow \infty$ in (5.3), we find out that

$$-\theta - \langle v, \zeta \rangle - \ell \geq 0, \quad \forall (\theta, \zeta) \in \partial_V \vartheta(t, x). \quad (5.4)$$

Notice that because $(v, \ell) \in \Gamma_i(x)$ for some $i \in \mathcal{I}_j$ (which depends exclusively on (t, x)), we have $v = f_i(x, a)$ and $\mathcal{L}_i(x, a) \leq \ell$ for some $a \in \mathcal{A}_i$. Therefore, replacing this in (5.4) and taking supremum over $a \in \mathcal{A}_i$ such that $f_i(x, a) \in \mathcal{T}_{\mathcal{M}_i}^B(x)$ we get (\mathbf{C}_2) .

To complete the proof we need to prove our assumption that $(v, \ell) \in \Gamma_i(x)$ and $v \in \mathcal{T}_{\mathcal{M}_i}^B(x)$, for some $i \in \mathcal{I}_j$. To see this we consider the two cases:

1. Suppose that for $\bar{n} \in \mathbb{N}$ large enough, $y(s) \in \mathcal{M}_i$ for any $s \in (t, t + h_{\bar{n}}]$; in the light of Lemma 4.3 this is the case if $j \notin \mathcal{J}_0$. Under these circumstances, we get that $(v, \ell) \in \Gamma_i(x)$. Indeed, take $\varepsilon > 0$ arbitrary, by the upper semicontinuity of Γ_i at x there is $n_\varepsilon \in \mathbb{N}$ with $n_\varepsilon \geq \bar{n}$ so that

$$\Gamma_i(y(sh_n + t)) \subseteq \Gamma_i(x) + \mathcal{B}(0, \varepsilon), \quad \forall n \geq n_\varepsilon, \quad \forall s \in [0, 1]. \quad (5.5)$$

The choice of n_ε is independent of $s \in [0, 1]$, as a consequence of the Gronwall's Lemma; see for instance (4.7). Furthermore, by Remark 5.1, we have that $\Gamma_i(x) + \mathcal{B}(0, \varepsilon)$ is a compact convex set.

On the other hand, for any $n \in \mathbb{N}$ let us consider the measurable function $\gamma_n : [0, 1] \rightarrow \mathbb{R}^N \times \mathbb{R}$ defined via

$$\gamma_n(s) := (\dot{y}(sh_n + t), \dot{z}(sh_n + t)), \quad \text{a.e. } s \in [0, 1].$$

By Lemma 2.1 we have that $\gamma_n(s) \in \Gamma_i(y(sh_n + t))$ a.e. on $[0, 1]$. Furthermore, by the definition of γ_n , [30, Lemma 4.2] and (5.5) we have that

$$(v_n, \ell_n) = \int_0^1 \gamma_n(s) ds \in \Gamma_i(x) + \mathcal{B}(0, \varepsilon), \quad \forall n \geq n_\varepsilon.$$

Letting $n \rightarrow +\infty$ we find out that $(v, \ell) \in \Gamma_i(x) + \mathcal{B}(0, \varepsilon)$. Moreover, since $\varepsilon > 0$ is arbitrary, we get that $(v, \ell) \in \Gamma_i(x)$ because the images of Γ_i are compact thanks to (H_1) . We finally notice that, since $y(t + h_n) \rightarrow x$ as $n \rightarrow +\infty$ and $y(t + h_n) \in \mathcal{M}_i$ for all $n \in \mathbb{N}$, we have as well that $v \in \mathcal{T}_{\mathcal{M}_i}^B(x)$. Hence the conclusions follows.

2. Suppose that there is no $\delta > 0$ and $i \in \mathcal{I}_j$ for which $y(s) \in \mathcal{M}_i$ for any $s \in (t, t + \delta)$. We claim that in this case $(v, \ell) \in \Gamma(x)$ and $v \in \mathcal{T}_{\Upsilon_j}(x)$. By Lemma 4.3 this situation can only happen if $i \in \mathcal{J}_0$, and so by (H_2) , we have $\Gamma(x) \cap \mathcal{T}_{\Upsilon_j}(x) \times \mathbb{R} = \overline{\text{co}}(\Gamma(x)) \cap \mathcal{T}_{\Upsilon_j}(x) \times \mathbb{R}$.

By the network-like structure, we can take the sequence $\{h_n\}$ so that $y(t + h_n) \in \Upsilon_j$ for any $n \in \mathbb{N}$, and this implies that $v \in \mathcal{T}_{\Upsilon_j}^B(x) = \mathcal{T}_{\Upsilon_j}(x)$. Moreover, using the same argument as in the preceding part, we can show by the upper semicontinuity of Γ at x together with [30, Lemma 4.2] that for any $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ so that

$$(v_n, \ell_n) = \int_0^1 \gamma_n(s) ds \in \overline{\text{co}}(\Gamma(x) + \mathcal{B}(0, \varepsilon)), \quad \forall n \geq n_\varepsilon.$$

Recall that $\Gamma(x)$ is not necessarily convex, so the righthand side cannot be replaced with the smaller set $\Gamma(x) + \mathcal{B}(0, \varepsilon)$. Repeating the reasoning of the case studied before, we can check that $(v, \ell) \in \overline{\text{co}}(\Gamma(x))$. However, by (H_2) the claim holds true because

$$(v, \ell) \in \overline{\text{co}}(\Gamma(x)) \cap \mathcal{T}_{\Upsilon_j}(x) \times \mathbb{R} = \Gamma(x) \cap \mathcal{T}_{\Upsilon_j}(x) \times \mathbb{R}.$$

Furthermore, by the definition of $\Gamma(x)$ and given that $\mathcal{T}_{\Upsilon_j}(x) \subseteq \mathcal{T}_{\mathcal{M}_i}^B(x)$ for any $i \in \mathcal{I}$, there exists indeed a particular $i \in \mathcal{I}_j$ so that $(v, \ell) \in \Gamma_i(x)$ with $v \in \mathcal{T}_{\mathcal{M}_i}^B(x)$. Therefore, the proof is complete. \square

Lemma 5.2. *Under the assumptions of Theorem 3.1, any $\omega : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ bilateral viscosity solution to (HJ) that verifies the junction conditions (C_0) and (C_2) , must satisfy in addition*

$$\vartheta(t, x) \leq \omega(t, x), \quad \forall (t, x) \in [0, T] \times \mathcal{K}.$$

Proof. We argue by contradiction, that is, assume there is $(t_0, x_0) \in [0, T] \times \mathcal{K}$ so that $\omega(t_0, x_0) < \vartheta(t_0, x_0)$. By (C_0) we can immediately rule out the case $t_0 = T$.

For sake of clarity, we divide the rest of the proof into three steps, each one of them aiming to prove a distinct claim.

Step 1: We first study the case in which x_0 belong to one of the branches of the network.

Claim A: If $x_0 \in \mathcal{M}_i$ for some $i \in \mathcal{I}$, then there exist $\tau > t_0$ and $y \in \mathbb{S}^\tau(t_0, x_0)$ so that

$$y(t) \in \mathcal{M}_i, \quad \forall t \in [t_0, \tau] \quad \text{and} \quad \omega(t, y(t)) < \vartheta(t, y(t)), \quad \forall t \in [t_0, \min\{\tau, T\}].$$

Furthermore, if $\tau \leq T$ then $y(\tau) \in \overline{\mathcal{M}_i} \setminus \mathcal{M}_i$.

To prove this, we only need to show that there exist $\tau \in (t_0, T)$ and $y \in \mathbb{S}^T(t_0, x_0)$ that remains on \mathcal{M}_i on (t_0, τ) and that verifies

$$\omega(t, y(t)) + \int_{t_0}^t \mathcal{L}(y(s), \dot{y}(s)) ds \leq \omega(t_0, x_0), \quad \forall t \in [t_0, \tau]. \quad (5.6)$$

Indeed, by the contradiction hypothesis and the Dynamic Programming Principle the conclusion follows easily. Now to prove (5.6), we use a weak invariance argument on the branch \mathcal{M}_i similar to the one used in [17, 18]. Before going further we recall the notion of proximal normal cone. Given a locally closed set $S \subseteq \mathbb{R}^n$, vector $\eta \in \mathbb{R}^n$ is called proximal normal to S at $x \in S$ if there exists $\sigma = \sigma(x, \eta) > 0$ so that

$$|\eta| \|x - \tilde{x}\|^2 \geq 2\sigma \langle \eta, \tilde{x} - x \rangle, \quad \forall \tilde{x} \in S.$$

The set of all such vectors η is the *Proximal normal cone* to S at x and which we denote by $\mathcal{N}_S^P(x)$.

Let us begin by noticing that the dynamics Γ_i can be extended to a multifunction defined on \mathbb{R}^N . Indeed, this is a consequence of the Extension Theorem (cf. [30, Theorem 2.6]) and its construction is as follows. Let $r > 0$ so that for each $y \in \mathbb{S}^T(t_0, x_0)$ we have $y(t) \in \mathbb{B}(x_0, r)$ for any $t \in [t_0, T]$; the existence of such $r > 0$ is given by Gronwall's Lemma. Let $\mathcal{O}_i \subseteq \mathbb{R}^N$ be the largest open subset contained in $\mathbb{B}(x_0, r)$ so that $\mathcal{K} \cap \mathcal{O}_i \subseteq \mathcal{M}_i$; the existence of \mathcal{O}_i is justified by the fact that the set of branches is locally finite in space. Hence, by [30, Theorem 2.6] there is a u.s.c. map with nonempty convex compact images defined on \mathbb{R}^N which agrees with Γ_i on $\overline{\mathcal{M}_i} \cap \overline{\mathcal{O}_i}$. We denote by $\hat{\Gamma}_i$ such an extension.

Let us introduce the l.s.c. function $W : [0, T] \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ defined via

$$W(t, x, z) := \omega(t, x) + z, \quad \forall (t, x, z) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}.$$

Note that $\partial_P \omega(t, x) \times \{1\} \subseteq \partial_P W(t, x, z)$ for any $(t, x, z) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}$. Hence, by [13, Theorem 11.31] we get that

$$(\theta, \zeta) \in \partial_P \omega(t, x), \quad (t, x) \in [0, T] \times \mathcal{K} \quad \implies \quad (\theta, \zeta, 1, -1) \in \mathcal{N}_{\text{epi}(W)}^P(t, x, z, W(t, x, z)), \quad \forall z \in \mathbb{R}.$$

Therefore, with the help of standard arguments in nonsmooth analysis, which we deliberately skip but refer to [14, Theorem 4.5.7] or [18, Proposition 5.1] for more details, we can prove that (2.1) implies that

$$\min_{(v, \ell) \in \Gamma_i(x)} \langle (1, v, \ell, 0), \eta \rangle \leq 0 \quad \forall \eta \in \mathcal{N}_{\text{epi}(W)}^P(t, x, z, W(t, x, z)), \quad (t, x, z) \in (0, T) \times \mathcal{M}_i \cap \mathcal{O}_i \times \mathbb{R}. \quad (5.7)$$

Let $\mathcal{U}_i := (0, T) \times \mathcal{O}_i \times \mathbb{R} \times \mathbb{R}$, then the map $x \mapsto \{1\} \times \hat{\Gamma}_i(x) \times \{0\}$ is u.s.c. with locally bounded images and it has nonempty compact convex images on $\text{epi}(W) \cap \mathcal{U}_i$. Consequently, all the conditions are met to apply the weak invariance criterion of [33]. Thus, [33, Theorem 3.1 (a)] combined with (5.7) implies that the set $\text{epi}(W)$ is weakly invariant in \mathcal{U}_i for the dynamics $\{1\} \times \hat{\Gamma}_i(\cdot) \times \{0\}$ (see [33, Definition 3.1]), which means in particular that, since $(t_0, x_0, 0, \omega(t_0, x_0)) \in \text{epi}(W)$, we can find $\tau > t_0$ and a curve $\gamma : [t_0, \tau] \rightarrow \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$ that solves $\dot{\gamma} \in \{1\} \times \hat{\Gamma}_i(\gamma) \times \{0\}$ a.e. on $[t_0, \tau]$, that lives in $\text{epi}(W)$ on $[t_0, \tau]$ and verify $\gamma(t_0) = (t_0, x_0, 0, \omega(t_0, x_0))$. It is rather clear that

$$\gamma(t) = (t, y(t), z(t), \omega(t_0, x_0)), \quad \forall t \in [t_0, \tau]$$

where $y \in \mathbb{S}^\tau(t_0, x_0)$ and $\dot{z}(t) \geq \mathcal{L}(y(t), \dot{y}(t))$ for a.e. $t \in [t_0, \tau]$. Furthermore, since $\gamma(t) \in \text{epi}(W) \cap \mathcal{U}_i$ for any $t \in [t_0, \tau]$, we have that $y(t) \in \mathcal{M}_i$ and $\omega(t, y(t)) + z(t) \leq \omega(t_0, x_0)$ for each $t \in [t_0, \tau]$. Hence, if $\tau > T$, the conclusion follows easily, so let us assume that it is not the case. By Gronwall's inequality (4.7), we see

that $y(\tau)$ is well defined and $y(t) \rightarrow y(\tau)$ as $t \rightarrow \tau^-$. Therefore, by the lower semicontinuity of ω we get that the (5.6) holds true and so does the first part of Claim A. The fact that if $\tau < T$ then $y(\tau) \in \overline{\mathcal{M}_i} \setminus \mathcal{M}_i$ comes from the value of τ given by [33, Theorem 3.1 (a)]. As a matter of fact, $\tau = \inf\{t > t_0 \mid \gamma(t) \notin \mathcal{O}_i\}$. Since $y(\tau) \in \mathbb{B}(x_0, r)$, the only option is that $y(\tau) \notin \mathcal{M}_i$. So the proof of Claim A is complete.

On the other hand, let $\tau \in (t, T]$ and $y \in \mathbb{S}^\tau(t_0, x_0)$ be as in Claim A. Note that if $\tau \geq T$ we can evaluate (5.6) at $\tau = T$ and use (C₀) to get

$$\psi(y(T)) + \int_{t_0}^T \mathcal{L}(y(s), \dot{y}(s)) ds = \omega(T, y(T)) + \int_{t_0}^T \mathcal{L}(y(s), \dot{y}(s)) ds \leq \omega(t_0, x_0) < \vartheta(t_0, x_0),$$

which contradicts the definition of the Value Function. So, the only interesting case that remains is when $\tau < T$. In this case we have $y(\tau) \in \overline{\mathcal{M}_i} \setminus \mathcal{M}_i$, and thus, by the network structure of \mathcal{K} , we have that there is $j \in \mathcal{J}$ so that $y(\tau) \in \Upsilon_j$ with $\omega(\tau, y(\tau)) < \vartheta(\tau, y(\tau))$. Therefore, setting $t_1 = \tau$ and $x_1 = y(\tau)$ we see that contradiction hypothesis yields to a new contradiction hypothesis, namely, $\omega(t_1, x_1) < \vartheta(t_1, x_1)$ for some $t_1 < T$ and $x_1 \in \Upsilon_j$ for some $j \in \mathcal{J}$. We study this case in the next steps.

Step 2: As disclosed in the preceding step, we only need to focus on the case that the inequality $\vartheta \leq \omega$ fails at some point on a junction. To do so, we investigate separately the case $j \in \mathcal{J}_0$ and $j \notin \mathcal{J}_0$.

Suppose there are $j \in \mathcal{J}_0$, $t_1 < T$ and $x_1 \in \Upsilon_j$ so that $\omega(t_1, x_1) < \vartheta(t_1, x_1)$. Under these circumstances a similar result as Claim A can be stated. Its proof is rather similar and it is a consequence of the fact that the dynamics of the convexified and original problem coincide around the junctions indexed by \mathcal{J}_0 .

Claim B: If $x_1 \in \Upsilon_j$ for some $i \in \mathcal{J}_0$, then there exist $\tau > t_1$ and $y \in \mathbb{S}^\tau(t_1, x_1)$ so that

$$\omega(t, y(t)) < \vartheta(t, y(t)), \quad \forall t \in [t_1, \tau].$$

Furthermore, if $\tau \leq T$ then $y(\tau) \in \Upsilon_l$ for some $l \in \mathcal{J} \setminus \{j\}$.

In a similar way as for Step 1, choose $r > 0$ so that for each $y \in \mathbb{S}^T(t_1, x_1)$ we have $y(t) \in \mathbb{B}(x_1, r)$ for any $t \in [t_1, T]$ and consider $\mathcal{O}_j \subseteq \mathbb{R}^N$ the largest open subset of $\mathbb{B}(x_1, r)$ such that Υ_j is the unique junction of \mathcal{K} intersecting \mathcal{O}_j . The idea is to prove that (C₂) and (2.1) imply that $\text{epi}(W)$ is weakly invariant in $\mathcal{U}_j := (0, T) \times \mathcal{O}_j \times \mathbb{R} \times \mathbb{R}$ for a suitable dynamics, which is an extension of the map $x \mapsto \{1\} \times \text{co}(\Gamma(x)) \times \{0\}$. By showing this and proceeding as for Claim A, the conclusion will be easily reached.

Recall that $x \mapsto \text{co}(\Gamma(x))$ is u.s.c. on \mathcal{K} . Hence, in the light of the Extension Theorem (cf. [30, Theorem 2.6]) and similarly as done earlier, we can construct a u.s.c. set valued-map with nonempty compact convex images, defined all along \mathbb{R}^N and such that it agrees with the map $x \mapsto \{1\} \times \text{co}(\Gamma(x)) \times \{0\}$ on $\overline{\mathcal{O}_j}$. We write F for such extension.

Note that F agrees with $\{1\} \times \Gamma_i \times \{0\}$ on $\mathcal{M}_i \cap \mathcal{O}_j$ for each $i \in \mathcal{I}_j$. Thus, the same arguments used for Claim A yield

$$\min_{\nu \in F(x)} \langle \nu, \eta \rangle \leq 0, \quad \forall \eta \in \mathcal{N}_{\text{epi}(W)}^P(t, x, z, W(t, x, z)), \quad \forall i \in \mathcal{I}_j, \quad \forall (t, x, z) \in (0, T) \times \mathcal{M}_i \cap \mathcal{O}_j \times \mathbb{R}.$$

Consequently, to provide the weak invariance of the system, it only remains to show that (C₂) implies that

$$\min_{\nu \in F(x)} \langle \nu, \eta \rangle \leq 0, \quad \forall \eta \in \mathcal{N}_{\text{epi}(W)}^P(t, x, z, W(t, x, z)), \quad \forall (t, x, z) \in (0, T) \times \Upsilon_j \cap \mathcal{O}_j \times \mathbb{R}. \quad (5.8)$$

To see this, it is enough to note that given that $\text{co}(\Gamma)$ has compact images, we obtain that

$$\min_{\nu \in F(x)} \langle \nu, \eta \rangle = \min_{(v, \ell) \in \text{co}(\Gamma(x))} \langle (1, v, \ell, 0), \eta \rangle = \min_{(v, \ell) \in \Gamma(x)} \langle (1, v, \ell, 0), \eta \rangle \quad \forall (x, \eta) \in \Upsilon_j \times \mathbb{R}^{N+3}.$$

Let $\eta = (\theta, \zeta, 1, -1)$ for some $\theta \in \mathbb{R}$ and $\zeta \in \mathbb{R}^N$, then from the last identities we get

$$\min_{\nu \in F(x)} \langle \nu, (\theta, \zeta, 1, -1) \rangle = \theta - \max_{i \in \mathcal{I}_j} H_i(x, \zeta) \leq \theta - H_i^+(x, \zeta), \quad \forall i \in \mathcal{I}_j, \quad \forall (x, \theta, \zeta) \in \Upsilon_j \times \mathbb{R} \times \mathbb{R}^N.$$

Thus, the standard arguments in nonsmooth analysis mentioned in step 1 we can prove that (5.8) holds. So, by [33, Theorem 3.1 (a)] we have that $\text{epi}(W)$ is weakly invariant in \mathcal{U} for the dynamics F .

In particular, by the same reasoning of the preceding claim, there exist $\tau > t_1$, an absolutely continuous curve $y : [t_1, \tau] \rightarrow \mathcal{K}$ with $y(t_1) = x_1$ and an absolutely continuous function $z : [t_1, \tau] \rightarrow \mathbb{R}$ with $z(t_1) = 0$, for which

$$\omega(t, y(t)) + z(t) \leq \omega(t_1, x_1), \quad \forall t \in [t_1, \tau), \quad \text{and} \quad (\dot{y}(t), \dot{z}(t)) \in \text{co}(\Gamma(y(t))), \quad \text{for a.e. } t \in [t_1, \tau).$$

Moreover, we can easily see that

$$\dot{y}(t) \in \begin{cases} F_i(y(t)) & \text{whenever } y(t) \in \mathcal{M}_i, \\ \mathcal{T}_{\Upsilon_j}(y(t)) & \text{whenever } y(t) \in \Upsilon_j, \end{cases} \quad \text{for a.e. } t \in [t_1, \tau).$$

But, by (H₂), since $j \in \mathcal{J}_0$ we have that $\dot{y}(t) \in F_j(y(t))$ for a.e. $t \in [t_1, \tau)$ so that $y(t) \in \Upsilon_j$. This remark implies that $y(\cdot)$ is a trajectory of the original control system, and so the first part of Claim B holds true. For the last part of Claim B we need to slightly modify the arguments used for Claim A. Indeed, by [33, Theorem 3.1 (a)] the value of τ is given by $\inf\{t > t_0 \mid \gamma(t) \notin \mathcal{O}_j\}$, which means that if $\tau \leq T$, then for some $i \in \mathcal{I}_j$ we must have $y(\tau) \in \overline{\mathcal{M}}_i \setminus \Upsilon_j$. Therefore, the proof of Claim B is complete.

On the other hand, the same reasons used for the conclusion of Claim A show that, if x_1 is as in Claim B and $\tau > T$ we get a contradiction. Consequently, we may restrict our attention to the case $\tau \leq T$. By Claim B, we have that $y(\tau) \in \Upsilon_l$ for some $l \in \mathcal{J}$. If $l \in \mathcal{J}_0$, we can use Claim B with $j = l$, $t_1 = \tau$ and $x_1 = y(\tau)$ to find another $\bar{\tau} > \tau$ and $\bar{y} \in \mathbb{S}^{\bar{\tau}}(\tau, y(\tau))$ and so that $y(\bar{\tau}) \in \Upsilon_n$ for some $n \neq j$. It is clear that we can repeat the argument to find $\bar{\tau} > t_1$ and $\bar{y} \in \mathbb{S}^{\bar{\tau}}(t_1, x_1)$ so that, either

$$\bar{\tau} > T \quad \text{and} \quad \omega(t, \bar{y}(t)) < \omega(t_1, x_1), \quad \forall t \in [t_1, \bar{\tau}],$$

or $\bar{y}(\bar{\tau}) \in \Upsilon_j$ for some $j \notin \mathcal{J}_0$. Since in the first case we find right away a contradiction with (C₀), it is evident that the only case that remains to dismiss is when $t_2 = \bar{\tau}$ and $x_2 = \bar{y}(\bar{\tau})$ so that $t_2 < T$ and $x_2 \in \Upsilon_j$ for $j \notin \mathcal{J}_0$.

Step 3: We now finally study the case described above, that is, we assume that for some $j \in \mathcal{J} \setminus \mathcal{J}_0$, there is $t_2 < T$ and $x_2 \in \Upsilon_j$ so that $\omega(t_2, x_2) < \vartheta(t_2, x_2)$. This situation is the last one we need to rule out in order to get a contradiction with the initial assumption that ω is not smaller or equal than the Value Function. In this step we prove the analogous of Claim B but for the case $j \notin \mathcal{J}_0$.

Claim C: If $x_2 \in \Upsilon_j$ for some $i \notin \mathcal{J}_0$, then there exist $\tau > t_2$ and $y \in \mathbb{S}^\tau(t_2, x_2)$ so that

$$\omega(t, y(t)) < \vartheta(t, y(t)), \quad \forall t \in [t_2, \tau].$$

Furthermore, if $\tau \leq T$ then $y(\tau) \in \Upsilon_l$ for some $l \in \mathcal{J} \setminus \{j\}$.

Let $i \in \mathcal{I}_j$ be given by (C₂) for $(t_2, x_2) \in (0, T) \times \Upsilon_j$. Note that, since $\text{dom } F_j = \emptyset$ we must have that $f_i(x_2, \mathcal{A}_i) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}^C(x_2) \neq \emptyset$, which means in the light of Lemma 4.2, that $f_i(x, \mathcal{A}_i) \subseteq r\text{-int}(\mathcal{T}_{\overline{\mathcal{M}}_i}^C(x))$ for any $x \in \Upsilon_j$. Let \mathcal{O}_j be an open subset of \mathbb{R}^N defined in the same way as in Step 2. We show first that

$$-\theta + \max_{(v, \ell) \in \Gamma_i(x)} \langle -v, \zeta \rangle - \ell \geq 0, \quad \forall (t, x) \in (0, T) \times \overline{\mathcal{M}}_i \cap \mathcal{O}_j, \quad \forall (\theta, \zeta) \in \partial_P \omega_i(t, x), \quad (5.9)$$

where $\omega_i = \omega$ on $[0, T] \times \overline{\mathcal{M}}_i$ and is $+\infty$ elsewhere. To do this we only need to focus on $(t, x) \in (0, T) \times \Upsilon_j$ because (5.9) on $[0, T] \times \mathcal{M}_i$ is a direct consequence of (2.1). Hence, let us fix $(t, x) \in (0, T) \times \Upsilon_j$ and $(\theta, \zeta) \in \partial_P \omega_i(t, x)$. By the Sum Rule for the proximal subdifferential (see for instance [14, Theorem 1.8.3]), we can construct the following sequence:

- $\{(t_n, x_n)\} \in (0, T) \times \mathcal{K}$ with $(t_n, x_n) \rightarrow (t, x)$ and $\omega(t_n, x_n) \rightarrow \omega(t, x)$.
- $\{(\theta_n, \zeta_n)\} \in \mathbb{R} \times \mathbb{R}^N$ with $(\theta_n, \zeta_n) \in \partial_P \omega(t_n, x_n)$ for any $n \in \mathbb{N}$.

- $\{(\tilde{x}_n, \eta_n)\} \in \overline{\mathcal{M}}_i \times \mathbb{R}^N$ with $\eta_n \in \mathcal{N}_{\overline{\mathcal{M}}_i}^P(\tilde{x}_n)$ for any $n \in \mathbb{N}$.

Furthermore, these sequences also verify that $\theta_n \rightarrow \theta$ and $\zeta_n + \eta_n \rightarrow \zeta$ as $n \rightarrow +\infty$. Suppose that there exists a subsequence of $\{x_n\}$ that lies in \mathcal{M}_i , then, avoiding relabelling the subsequence, (2.1) implies that

$$-\theta_n + \max_{(v, \ell) \in \Gamma_i(x_n)} \langle -v, \zeta_n \rangle - \ell \geq 0, \quad \forall n \in \mathbb{N}.$$

Since Γ_i has compact images, for each $n \in \mathbb{N}$ there exists $(\ell_n, v_n) \in \Gamma_i(x_n)$ so that $\theta_n + \langle v_n, \zeta_n \rangle + \ell_n \leq 0$. Moreover, since Γ_i is u.s.c. and uniformly bounded around x , we can assume that $\{(v_n, \ell_n)\}$ converges to some $(v, \ell) \in \Gamma_i(x)$. Additionally, since f_i is continuous and $f_i(x, \mathcal{A}_i) \subseteq r\text{-int}(\mathcal{T}_{\overline{\mathcal{M}}_i}^C(x))$, there exists $\{\varepsilon_n\} \subseteq (0, 1)$ so that $\varepsilon_n \rightarrow 0$ and $\langle v_n, \eta_n \rangle \leq \varepsilon_n$. So, gathering the information we find out that

$$\theta_n + \langle v_n, \zeta_n + \eta_n \rangle + \ell_n \leq \varepsilon_n, \quad \forall n \in \mathbb{N}.$$

Letting $n \rightarrow +\infty$, we get (5.9) after a few algebraic steps.

On the other hand, if there is no subsequence of $\{x_n\}$ lying in \mathcal{M}_i , we may assume that $x_n \in \Upsilon_j$ for any $n \in \mathbb{N}$. However, in this case (C₂) yields

$$-\theta_n + \max_{(v, \ell) \in \Gamma_i(\tilde{x})} \langle -v, \zeta_n \rangle - \ell \geq 0, \quad \forall n \in \mathbb{N}.$$

Hence, using the same arguments as above we can easily prove that (5.9) holds as well.

Now, let us introduce the l.s.c. function $W_i : [0, T] \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ defined via

$$W_i(t, x, z) := \omega_i(t, x) + z, \quad \forall (t, x, z) \in [0, T] \times \mathbb{R}^N \times \mathbb{R},$$

and consider a multifunction F_i , which is a u.s.c. extension of the set-valued map $x \mapsto \{1\} \times \Gamma_i(x) \times \{0\}$ from $\overline{\mathcal{M}}_i \cap \overline{\mathcal{O}}_j$ up to \mathbb{R}^N . Then, it is not difficult to see that (5.9) implies that

$$\min_{\nu \in F_i(x)} \langle w, \eta \rangle \leq 0 \quad \forall \eta \in \mathcal{N}_{\text{epi}(W_i)}^P(t, x, z), \quad \forall (t, x, z) \in (0, T) \times \overline{\mathcal{M}}_i \times \mathbb{R}.$$

Therefore, by [33, Theorem 3.1 (a)], the conclusion follows by the same arguments used in the previous claims. Consequently, Claim C has been proved.

Step 4: Finally, recall that the set of junctions is locally finite and pairwise disjoint. Therefore, for any $(t, x) \in (0, T) \times \mathcal{K}$ such that $\omega(t, x) < \vartheta(t, x)$, by concatenating the trajectories obtained by the preceding claims we can find $\tau \geq T$ and $y \in \mathbb{S}^\tau(t, x)$ so that

$$\omega(s, y(s)) < \vartheta(s, y(s)), \quad \forall s \in [t, \tau].$$

Evaluating the preceding inequality at $s = T$ we get a contradiction with (C₀). If $t = 0$, the result come from the lower semicontinuity of ω and the dynamic programming principle. Considering all these arguments, the proof of Lemma 5.2 is complete. \square

6 Maximality of the Value Function as subsolution

6.1 Technical lemmas on strong invariance

One of the key result required to prove Theorem 3.2 is based on strong invariance arguments, which is summarized below (Lemma 6.2). However, before going further it may be helpful to recall some notion of Variational Analysis, in particular of Proximal Analysis.

6.1.1 Normal Cones and Proximal subgradients.

For sake of the exposition, we recall the definition of the Proximal normal cone and its relation with the proximal subgradient. For a further discussion about this topic we refer the reader to [14].

Let $\mathcal{S} \subseteq \mathbb{R}^k$ be a locally closed set and $x \in \mathcal{S}$. Recall that $\mathcal{N}_{\mathcal{S}}^P(x)$ stands for the *Proximal normal cone* to \mathcal{S} at x . If $\mathcal{S} = \text{epi}(\omega)$ where $\omega : \mathbb{R}^k \rightarrow \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function, then for every $x \in \text{dom}(\omega)$, the following relation is valid:

$$\partial_P \omega(x) \times \{-1\} \subseteq \mathcal{N}_{\text{epi}(\omega)}^P(x, \omega(x)), \quad \forall x \in \text{dom}(\omega).$$

By definition of the proximal subdifferential, $\zeta \in \partial_P \varphi(x)$ if and only if there exist $\sigma, \delta > 0$ such that

$$\varphi(y) \geq \varphi(x) + \langle \zeta, y - x \rangle - \sigma |y - x|^2 \quad \forall y \in \mathcal{B}(x, \delta) \cap \text{dom} \varphi.$$

This inequality is called the *proximal subgradient inequality*.

6.1.2 HJB equations and invariance

Lemma 6.1. *Let M be a manifold, let $\mathbb{F} : M \rightarrow \mathbb{R}^N$ be set-valued map with nonempty images such that $F(x) \subseteq \mathcal{T}_M(x)$ for any $x \in M$ and let $\mathbb{L} : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a continuous function, such that the following multifunction is locally Lipschitz on M :*

$$x \mapsto \{(v, \ell) \mid v \in \mathbb{F}(x), \ell = \mathbb{L}(x, v)\}. \quad (6.1)$$

Let $\omega : [0, T] \times M \rightarrow \mathbb{R}$ be a lower semicontinuous function and suppose that:

$$-\theta + \sup_{v \in \mathbb{F}(x)} \{-\langle v, \zeta \rangle - \mathbb{L}(x, v)\} \leq 0, \quad \forall (t, x) \in (0, T) \times M, \quad \forall (\theta, \zeta) \in \partial_P \omega(t, x). \quad (6.2)$$

Then, for any $0 < a < b < T$ and $y : [a, b] \rightarrow M$ absolutely continuous arc that satisfies

$$\dot{y}(s) \in \mathbb{F}(y(s)), \quad \text{a.e. } s \in [a, b], \quad (6.3)$$

one has

$$\omega(a, y(a)) \leq \int_a^b \mathbb{L}(y(s), \dot{y}(s)) ds + \omega(b, y(b)). \quad (6.4)$$

Proof. Since M is a manifold (at least C^2), there is a tubular neighborhood around M , that is, there exists an open set $U \subseteq \mathbb{R}^N$ such that the projection over M is well defined on U and is (at least) a C^1 submersion on U ; see for instance [23, Theorem 6.24]. Hence, it is easy to see that \mathbb{F} and \mathbb{L} can be extended up to U in such a way that the multifunction given in (6.1) is locally Lipschitz on U as well. A similar remark is valid for the function ω and the HJB inequality (6.2), which we assume from now on that are defined and valid on $(0, T) \times U$, respectively.

On the other hand, to show that (6.4) holds it is enough to prove that $W : [0, T] \times U \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$W(t, x, z) := \omega(t, x) + z$$

is strongly increasing on $(0, T) \times U$ for the dynamics given by (6.1). Thus, the conclusion follows directly from [14, Proposition 4.6.5] and the fact that

$$\partial_P W(t, x, z) \subseteq \partial_P \omega(t, x) \times \{1\}$$

□

In the case that M is not a necessarily a manifold, a similar result can be stated. In particular, this is relevant for the case when M is a manifold with boundary.

Lemma 6.2. *Let M be a closed set, let $\mathbb{F} : M \rightarrow \mathbb{R}^N$ be set-valued map with nonempty images and let $\mathbb{L} : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a continuous function, such that the following multifunction is locally Lipschitz on M :*

$$x \mapsto \{(v, \ell) \mid v \in \mathbb{F}(x), \ell = \mathbb{L}(x, v)\}$$

Let $\omega : [0, T] \times M \rightarrow \mathbb{R}$ be a lower semicontinuous function and suppose that it satisfies (6.2). Then, the same conclusion as in Lemma 6.1 holds.

Proof. Let $a, b \in \mathbb{R}$ and $y : [a, b] \rightarrow M$ be an absolutely continuous curve that satisfies (6.3). Let us consider the curve $\tilde{y} : [a, b] \rightarrow M$ given by $\tilde{y}(s) = y(a + b - s)$ for any $s \in [a, b]$. Let $W : [0, T] \times M \times \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$W(t, x, z) := \omega(t, x) + z$$

Hence, defining

$$\gamma(t) = \left(a + b - t, \tilde{y}(t), - \int_a^t \mathbb{L}(\tilde{y}(s), -\dot{\tilde{y}}(s)) ds, \omega(b, y(b)) \right), \quad \forall t \in [a, b],$$

we see that if $\gamma(t) \in \text{epi}(W)$ for every $t \in [a, b]$, the conclusion of the lemma follows immediately. Furthermore,

$$\dot{\gamma}(t) \in \Gamma(\gamma(t)), \quad \text{a.e. } t \in [a, b], \quad (6.5)$$

where

$$\Gamma(t, x, z) = \{-1\} \times \{(-v, -\ell) \mid v \in \mathbb{F}(x), \ell = \mathbb{L}(x, v)\} \times \{0\}.$$

It's clear that the condition on the epigraph of W holds at $t = a$, that is, $\gamma(a) \in \text{epi}(W)$. Hence, such condition amounts to say that $\text{epi}(W)$ is strongly invariant for trajectories that satisfies (6.5) on the interval $[a, b]$. Therefore, to conclude we need to apply some ad hoc criterion for strong invariance, as for example [18, Proposition 4.2].

To do so, we proceed as follows. We set $\mathcal{S} = \text{epi}(W)$ and $\mathbb{M} = \mathbb{R} \times M \times \mathbb{R} \times \mathbb{R}$, and take $R > 0$ fixed. Note that $\mathcal{S} \subseteq \mathbb{M}$ is also closed, and Γ is locally Lipschitz on \mathbb{M} , which means that there is $L_\Gamma > 0$ such that

$$\text{dist}_{\Gamma(p)}(q) \leq L_\Gamma |p - \tilde{p}|, \quad \forall p, \tilde{p} \in \mathcal{B}(0, \tilde{R}) \cap \overline{\mathbb{M}}, \quad q \in \Gamma(\tilde{p}),$$

where $\tilde{R} > R$ so that $\text{proj}_{\mathcal{S}}(p) \subseteq \mathcal{B}(0, \tilde{R})$ for any $p \in \mathcal{B}(0, R)$.

Let $p \in \mathcal{B}(0, R) \cap \mathbb{M}$ and $s \in \text{proj}_{\mathcal{S}}(p)$, then for any $q \in \Gamma(p)$ and $\varepsilon > 0$, there is $q_s^\varepsilon \in \Gamma(s)$ such that

$$\langle p - s, q \rangle = \langle p - s, q - q_s^\varepsilon \rangle + \langle p - s, q_s^\varepsilon \rangle \leq |p - s| (\text{dist}_{\Gamma(s)}(q) + \varepsilon) + \langle p - s, q_s^\varepsilon \rangle.$$

Thus, in the light of the Lipschitz estimate for Γ we have that

$$\langle p - s, q \rangle \leq L_\Gamma |p - s|^2 + \varepsilon |p - s| + \langle p - s, q_s^\varepsilon \rangle. \quad (6.6)$$

The next step consists in showing that

$$\langle p - s, q_s^\varepsilon \rangle \leq 0.$$

Indeed, note that $p - s \in \mathcal{N}_{\mathcal{S}}^P(s)$ and $s = (t_s, x_s, z_s, W(t_s, x_s, z_s))$, and since \mathcal{S} is the epigraph of a function we have that $p - s = (\xi, -\lambda)$, where $\lambda \geq 0$. We study first the case $\lambda > 0$. Under these circumstances we have that

$$\frac{1}{\lambda} \xi \in \partial_P W(t_s, x_s, z_s) \subseteq \partial_P \omega(t_s, x_s) \times \{1\}.$$

In other words, $\frac{1}{\lambda} \xi = (\theta, \zeta, 1)$ for some $(\theta, \zeta) \in \partial_P \omega(t_s, x_s)$. Therefore, since $q_s^\varepsilon = (-1, -v_s, -\mathbb{L}(s, v_s), 0)$ for some $v_s \in \mathbb{F}(x_s)$ we have

$$\langle p - s, q_s^\varepsilon \rangle = \lambda(-\theta - \langle v_s, \zeta \rangle - \mathbb{L}(x_s, v_s)) \leq \lambda(-\theta + \sup_{v \in \mathbb{F}(x_s)} \{-\langle v, \zeta \rangle - \mathbb{L}(x_s, v)\}) \leq 0$$

If on the other hand, $\lambda = 0$, then $(\xi, 0) \in \mathcal{N}_S^P(t_s, x_s, z_s, W(t_s, x_s, z_s))$ and by Rockafellar's horizontality theorem (see for instance [28]), there exist some sequences $\{(\tau_n, x_n, z_n)\} \subseteq \text{dom}(W)$, $\{\xi_n\} \subseteq \mathbb{R}^{N+2}$ and $\{\lambda_n\} \subseteq (0, \infty)$ such that

$$\begin{aligned} (t_n, x_n, z_n) &\rightarrow (t_s, x_s, z_s), & W(t_n, x_n, z_n) &\rightarrow W(t_s, x_s, z_s), \\ (\xi_n, \lambda_n) &\rightarrow (\xi, 0) = p - s, & \frac{1}{\lambda_n} \xi_n &\in \partial_P W(t_n, x_n, z_n). \end{aligned}$$

Thus, using the same argument as above we can show

$$\langle (\xi_n, \lambda_n), q_s^n \rangle \leq 0, \quad \forall n \in \mathbb{N}, \quad \forall q_s^n \in \Gamma(t_n, x_n, z_n, W(t_n, x_n, z_n)).$$

Hence, due to the fact that Γ is locally Lipschitz continuous, there exists $q_s^n \in \Gamma(t_n, x_n, z_n, W(t_n, x_n, z_n))$ so that $q_s^n \rightarrow q_s^\varepsilon$. Therefore, we can pass into the limit in the preceding inequality, and taking into consideration (6.6), we obtain

$$\langle p - s, q \rangle \leq L_\Gamma |p - s|^2 + \varepsilon |p - s|.$$

Finally, since $\varepsilon > 0$ is arbitrary, the conclusion follows from [18, Proposition 4.2]. \square

6.2 Junctions condition of subsolution-type

Let us now check that the Value Function satisfies the junction condition **(C₃)**.

Lemma 6.3. *Let \mathcal{K} be a d -dimensional network and consider a family of control spaces $\{\mathcal{A}_i\}_{i \in \mathcal{I}}$ so that **(H_A)** holds. Let $\{\psi_i\}_{i \in \mathcal{I}}$, $\{\mathcal{L}_i\}_{i \in \mathcal{I}}$ and $\{f_i\}_{i \in \mathcal{I}}$ be collections of final costs, running costs and dynamics satisfying **(H_ψ)**, **(H_ℒ)** and **(H_f)**, respectively. Assume that **(H₁)**, **(H₀)**, **(H₂)** **(H₃)** are also verified. Then the Value Function of the Bolza problem on the d -dimensional network \mathcal{K} verifies the junction condition **(C₃)**.*

Proof. First of all, note that if $j \notin \mathcal{J}_0$ then $A_{ij}^0 = \emptyset$ and so **(C₃)** is equivalent to

$$\forall (t, x) \in (0, T] \times \Upsilon_j, \quad \forall i \in \mathcal{I}_j: \quad -\theta + H_i^-(x, \zeta) \leq 0, \quad \forall (\theta, \zeta) \in \partial_V \omega(t, x). \quad (6.7)$$

Therefore, this condition may be trivial if $A_{ij}^- = \emptyset$ as well; this is the case if trajectories starting at the junction Υ_j can only move away from Υ_j throughout \mathcal{M}_i .

Similarly, if $j \in \mathcal{J}_0$ and $A_{ij}^- = \emptyset$, then **(C₃)** is equivalent to

$$\forall (t, x) \in (0, T] \times \Upsilon_j, \quad \forall i \in \mathcal{I}_j: \quad -\theta + H_i^0(x, \zeta) \leq 0, \quad \forall (\theta, \zeta) \in \partial_V \omega(t, x). \quad (6.8)$$

The latter Hamiltonian is always finite because under these circumstances $A_{ij}^0 \neq \emptyset$.

To sum up, we only need to prove that (6.7) and (6.8) hold whenever appropriate, that is, when $A_{ij}^0 \neq \emptyset$ and $A_{ij}^- \neq \emptyset$, respectively.

Let $a \in \mathcal{A}_i$ and suppose that either $a \in \mathcal{A}_{ij}^-$ or $a \in \mathcal{A}_{ij}^0$. In the first instance we have, by **(H_f)** and **(H₃)** that

$$-f_i(\tilde{x}, a) \in r\text{-int} \left(\mathcal{T}_{\overline{\mathcal{M}}_i}^C(\tilde{x}) \right), \quad \forall \tilde{x} \in \overline{\mathcal{M}}_i.$$

In the second case, by **(H₃)** we have that

$$f_i(\tilde{x}, a) \in \mathcal{T}_{\Upsilon_j}(\tilde{x}), \quad \forall \tilde{x} \in \Upsilon_j.$$

Hence, the Nagumo's Theorem implies that there is $\delta > 0$ and a continuously differentiable arc $y : [t - \delta, t] \rightarrow \mathbb{R}^N$ such that

$$\dot{y}(s) = f_i(y(s), a), \quad y(s) \in \overline{\mathcal{M}}_i, \quad \forall s \in [t - \delta, t] \quad \text{and} \quad y(t) = x.$$

Note that if $a \in \mathcal{A}_{ij}^-$ we get that $y(s) \in \mathcal{M}_i$, $\forall s \in (t - \delta, t]$ and if $a \in \mathcal{A}_{ij}^0$, then $y(s) \in \Upsilon_j$, $\forall s \in [t - \delta, t]$.

Finally, picking up the arguments used in Proposition 2.2 (Step 2) and using the fact that f_i and \mathcal{L}_i are assumed to be continuous up to $\overline{\mathcal{M}_i} \times \mathcal{A}_i$ we get

$$0 \geq -\theta - \langle \zeta, f_i(x, a) \rangle - \mathcal{L}_i(x, a), \quad \forall (\theta, \zeta) \in \partial_V \vartheta(t, x).$$

Since a is an arbitrary element of \mathcal{A}_{ij}^- or \mathcal{A}_{ij}^0 , taking supremum over this variable we get either (6.7) or (6.7), when appropriate. □

6.3 Sufficiency of the junction conditions

We now prove that the junction conditions are indeed sufficient to single out the Value Function as unique bilateral viscosity solution to the HJB equation. To see this it is enough to prove the following result

Lemma 6.4. *Under the assumptions of Theorem 3.2, any $\omega : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ bilateral viscosity solution to (HJ) that verifies the junction condition (C₃), satisfies in addition*

$$\vartheta(t, x) \geq \omega(t, x), \quad \forall (t, x) \in [0, T] \times \mathcal{K}.$$

Proof. We can immediately discard the case $t = T$ because $\omega(T, x) = \psi(x) = \vartheta(T, x)$ for any $x \in \mathcal{K}$.

Let us assume by contradiction that there is $(t_0, x_0) \in [0, T] \times \mathcal{K}$ so that $\vartheta(t_0, x_0) < \omega(t_0, x_0)$. The proof consists in showing that for an optimal trajectory $y \in \mathbb{S}^T(t_0, x_0)$ that realizes the Value Function we have

$$\vartheta(t, y(t)) < \omega(t, y(t)), \quad \forall t \in [0, T],$$

which, in the light of (C₀), will yield to a contradiction with $\vartheta(T, y(T)) = \omega(T, y(T))$.

By the Dynamic Programming Principle, we have that any optimal trajectory verifies

$$\vartheta(t_0, x_0) = \vartheta(t, y(t)) + \int_{t_0}^t \mathcal{L}(y(s), \dot{y}(s)) ds, \quad \forall t \in [t_0, T]. \quad (6.9)$$

For sake of exposition, we divide the proof in several steps. We fix $y \in \mathbb{S}^T(t_0, x_0)$ to be an optimal trajectory.

Step 1: Let us start by considering the case that $x_0 \in \mathcal{M}_i$ for some $i \in \mathcal{I}$. We claim under these circumstances there exists $t_1 \in (t_0, T]$ such that $y(t_1) \in \overline{\mathcal{M}_i} \setminus \mathcal{M}_i$ and

$$\vartheta(t, y(t)) + \int_{t_0}^t \mathcal{L}(y(s), \dot{y}(s)) ds < \omega(t_0, x_0) \leq \omega(t, y(t)) + \int_{t_0}^t \mathcal{L}(y(s), \dot{y}(s)) ds, \quad \forall t \in [t_0, t_1] \quad (6.10)$$

Indeed, this is a consequence of Lemma 6.1 applied with $\mathbb{F} = f_i(\cdot, \mathcal{A}_i)$ and $\mathbb{L} = \mathcal{L}$. We only need to check that under our assumptions, the set-valued map

$$x \mapsto \{(v, \ell) \mid v \in f_i(x, \mathcal{A}_i), \ell = \mathcal{L}(x, v)\}$$

is locally Lipschitz. However, this is a direct consequence of the fact that $f_i(\cdot, a)$ and $\mathcal{L}_i(\cdot, a)$ are locally Lipschitz, uniformly for any $a \in \mathcal{A}_i$.

Step 2: Let us point out that if the function ω were continuous on $[0, T] \times \mathcal{K}$ (or continuous along optimal trajectories), then (6.10) would immediately imply

$$\vartheta(t_1, y(t_1)) < \omega(t_1, y(t_1)). \quad (6.11)$$

Nevertheless, in our bilateral approach, the function ω may only be lsc along an optimal trajectory and so, further developments are needed in order to prove (6.11).

Let us set $\mathbb{F}(x) = f_i(x, \mathcal{A}_{ij}^- \cup \mathcal{A}_{ij}^0)$. Note that this set-valued map is locally Lipschitz continuous, and similarly as in the preceding Step, it is easy to check that the following multifunction is locally Lipschitz continuous as well

$$x \mapsto \{(v, \ell) \mid v \in f_i(x, \mathcal{A}_{ij}^- \cup \mathcal{A}_{ij}^0), \ell = \mathcal{L}(x, v)\}.$$

If there were some $\delta > 0$ so that

$$\dot{y}(s) \in \mathbb{F}(y(s)), \quad \text{a.e. } s \in (\max\{t_0, t_1 - \delta\}, t_1] \quad (6.12)$$

then the conclusion would be a direct consequence of Lemma 6.2, applied with $M = \overline{\mathcal{M}}_i$. However, this is not necessarily true for any trajectory of the control systems, let alone optimal trajectories.

For this reason further arguments are required. In particular, we focus now on showing that any trajectory of the control systems can be approximated by a sequence of curves satisfying (6.12).

Let $\alpha : [t_0, t_1] \rightarrow \mathcal{A}_i$ be a measurable control given by Lemma 2.1, that is,

$$\dot{y}(t) = f_i(y(t), \alpha(t)), \quad \text{and} \quad \mathcal{L}(y(t), \dot{y}(t)) = \mathcal{L}_i(y(t), \alpha(t)) \quad \text{a.e. } t \in [t_0, t_1].$$

First of all, by (H₃), this situation can only occur if $\mathcal{A}_{ij}^- \neq \emptyset$, otherwise one would have $\mathcal{A}_i = \mathcal{A}_{ij}^0 \cup \mathcal{A}_{ij}^+$ and

$$h_{N-d+1}(y(t_1)) - h_{N-d+1}(y(t)) = \int_t^{t_1} \langle \nabla h_{N-d+1}(y(s)), f_i(y(s), \alpha(s)) \rangle ds, \quad \forall t \in [t_0, t_1],$$

with the left handside being positive and the right handside being non positive if t is close enough to t_1 , where $h : \mathbb{R}^{N-d+1} \rightarrow \mathbb{R}$ is a local defining map for $\overline{\mathcal{M}}_i$ around $y(t_1)$.

On the one hand, by Gronwall and the linear growth property of the dynamics f_i we have that

$$|y(t) - y(t_1)| \leq (e^{c_f(t_1-t)} - 1)(|y(t_1)| + 1), \quad \forall t \in [t_0, t_1].$$

This means that the graph of trajectory y on $[t_0, t_1]$ is contained in an open bounded set Ω . Let L_i be a common Lipschitz constant for $f_i(\cdot, \mathcal{A}_i)$ and $\mathcal{L}_i(\cdot, \mathcal{A}_i)$ on $\Omega + \mathbb{B}$.

On the other hand, since $\mathcal{A}_{ij}^- \neq \emptyset$, for any $a \in \mathcal{A}_{ij}^-$, we also get

$$|y_a(t) - y(t_1)| \leq (e^{c_f(t_1-t)} - 1)(|y(t_1)| + 1), \quad \forall t \in [\tau_a, t_1] \quad (6.13)$$

where $y_a : [\tau_a, t_1] \rightarrow \mathbb{R}^N$ is the maximal solution of

$$\dot{y}_a(t) = f_i(y_a(t), a), \quad \text{a.e. } t \in [\tau_a, t_1], \quad y_a(t_1) = y(t_1),$$

that lies on \mathcal{M}_i on $[\tau_a, t_1]$. Since $a \in \mathcal{A}_{ij}^-$, the existence of such trajectory is justified.

Let $\delta > 0$ such that $t_1 - \delta > \max\{t_0, \tau_a\}$. Combining the preceding two inequalities we obtain that

$$|y_a(t_1 - \delta) - y(t_1 - \delta)| \leq 2(e^{c_f\delta} - 1)(|y(t_1)| + 1). \quad (6.14)$$

Furthermore, by the Gronwall Lemma and the locally Lipschitz character of the dynamics f_i , we have that for some $L_i > 0$ (which depends only on $y(t_1)$ and $T > 0$)

$$|y(t) - y_\delta(t)| \leq e^{L_i(t_1-\delta-t)} |y(t_1 - \delta) - y_a(t_1 - \delta)|, \quad \forall t \in [\max\{t_0, \tau_a\}, t_1 - \delta] \quad (6.15)$$

where $y_\delta : [\tau_a, t_1 - \delta] \rightarrow \mathbb{R}^N$ is the maximal solution lying on \mathcal{M}_i of

$$\dot{y}_\delta(t) = f_i(y_\delta(t), \alpha(t)), \quad \text{a.e. } t \in (\tau_a, t_1 - \delta], \quad y_\delta(t_1 - \delta) = y_a(t_1 - \delta).$$

Thus, we have

$$|y(t) - y_\delta(t)| \leq 2(e^{c_f\delta} - 1)(|y(t_1)| + 1)e^{L_i(t_1-t_0)}, \quad \forall t \in [\max\{t_0, \tau_a\}, t_1 - \delta]$$

Hence, given $\varepsilon \in (0, 1)$ and taking $\delta > 0$ so that

$$2(e^{c_f\delta} - 1)(|y(t_1)| + 1)e^{L_i(t_1-t_0)} \leq \varepsilon,$$

if we set $t_\varepsilon = \max\{t_0, \tau_a\}$ and $x_\varepsilon = y_\delta(t_\varepsilon)$, we have that the curve

$$y_\varepsilon(t) = \begin{cases} y_\delta(t) & t \in [t_\varepsilon, t_1 - \delta], \\ y_a(t) & t \in [t_1 - \delta, t_1], \\ y(t) & t \in [t_1, T], \end{cases}$$

belongs to $\mathbb{S}^T(t_\varepsilon, x_\varepsilon)$, with $t_\varepsilon \rightarrow t_0$ and $x_\varepsilon \rightarrow x_0$ as $\varepsilon \rightarrow 0$. So, by the initial remark we have that

$$w(t_\varepsilon, x_\varepsilon) \leq \int_{t_\varepsilon}^{t_1 - \varepsilon} \mathcal{L}_i(y_\varepsilon(s), \alpha(s)) ds + \int_{t_1 - \varepsilon}^{t_1} \mathcal{L}_i(y_a(s), a) ds + w(t_1, y(t_1)). \quad (6.16)$$

Since $x \mapsto \mathcal{L}_i(x, a)$ is locally Lipschitz, uniformly with respect to $a \in \mathcal{A}_i$, we have, thanks to (6.15),

$$|\mathcal{L}_i(y(t), \alpha(t)) - \mathcal{L}_i(y_\varepsilon(t), \alpha(t))| \leq L_i |y(t) - y_\varepsilon(t)| \leq L_i \varepsilon, \quad \text{a.e. } t \in [t_\varepsilon, t_1 - \varepsilon].$$

Moreover, the continuity of \mathcal{L}_i and (6.13) imply that there is a constant $C > 0$ so that

$$0 \leq \mathcal{L}_i(y_a(t), a) \leq C, \quad \forall t \in [\tau_a, t_1].$$

Thus, we get

$$\int_{t_\varepsilon}^{t_1 - \varepsilon} \mathcal{L}_i(y_\varepsilon(s), \alpha(s)) ds + \int_{t_1 - \varepsilon}^{t_1} \mathcal{L}_i(y_a(s), a) ds \rightarrow \int_{t_0}^{t_1} \mathcal{L}_i(y(s), \alpha(s)) ds \quad \text{as } \varepsilon \rightarrow 0.$$

So, finally, using the lower semicontinuity of w we obtain from (6.16)

$$w(t_0, x_0) \leq \int_{t_0}^{t_1} \mathcal{L}_i(y(s), \alpha(s)) ds + w(t_1, y(t_1)),$$

which completes the proof of this step.

Step 3: The analysis we have done so far implies that, without loss of generality, we could have assumed from the very beginning that $x_0 \in \Upsilon_j$ for some $j \in \mathcal{J}$. Furthermore, the case $j \notin \mathcal{J}_0$ can be ruled out almost immediately. Indeed, if $j \notin \mathcal{J}_0$, then any trajectory of the control system can only pass through, without chattering, around Υ_j (see Lemma 4.3). Consequently, there is $i \in \mathcal{I}_j$ and $\delta > 0$ so that $y(s) \in \mathcal{M}_i$ for any $s \in (t_0, t_0 + \delta)$, which means that we fall in the framework studied in Step 1 and so by (6.10) we get

$$\omega(\tau, y(\tau)) \leq \omega(t, y(t)) + \int_\tau^t \mathcal{L}(y(s), \dot{y}(s)) ds, \quad \forall \tau, t \in (t_0, t_0 + \delta) \text{ with } \tau < t.$$

Then, the lower semicontinuity of ω implies that

$$\omega(t_0, x_0) \leq \omega(t, y(t)) + \int_{t_0}^t \mathcal{L}(y(s), \dot{y}(s)) ds, \quad \forall t \in [t_0, t_0 + \delta).$$

Thus, under these circumstances there would be $t_1 > t_0$ so that (using the arguments of Step 2 as well)

$$y(t_1) \in \overline{\mathcal{M}}_i \setminus \mathcal{M}_i, \quad \text{and} \quad \vartheta(t, y(t)) < \omega(t, y(t)), \quad \forall t \in [t_0, t_1].$$

Note that since the set of velocities of the control systems is bounded and the trajectory doesn't chatter around Υ_j , then it is possible to be bounded from below, in a uniform way, the difference $t_1 - t_0$. Thus, if the trajectory $t \mapsto y(t)$ never reaches a junction whose index belongs to \mathcal{J}_0 , we can repeat the process ad infinitum to eventually get

$$\vartheta(t, y(t)) < \omega(t, y(t)), \quad \forall t \in [t_0, T],$$

from where a contradiction can be reached by evaluating the inequality at $t = T$.

Step 4: In the light of the preceding comments, it only remains to study the case $x_0 \in \Upsilon_j$ with $j \in \mathcal{J}_0$. This situation is the most delicate to be analyzed, and for this reason we consider two different cases depending on the structure of trajectory y . This analysis uses techniques introduced in [18, 17].

Without loss of generality we assume that

$$y(T) \in \Upsilon_j \quad \text{and} \quad y(s) \in \Upsilon_j \cup \bigcup_{i \in \mathcal{I}_j} \mathcal{M}_i, \quad \forall t \in [t_0, T],$$

Otherwise, if the trajectory leaves the set $\Upsilon_j \cup \bigcup_{i \in \mathcal{I}_j} \mathcal{M}_i$ before the horizon time T , the trajectory will reach another (different) junction, from which the same the analysis can be done. Moreover, since the time to pass from one junction to another can always be bounded from below because the set of velocities is bounded, the process must finish in a finite number of iterations.

Non-chattering case: we might assume first that the set $\{s \in [t_0, T] \mid y(s) \in \Upsilon_j\}$ is made of a finite number of intervals closed intervals, that is, there are $t_0 = a_0 \leq b_0 < a_1 \leq b_1 < \dots < a_n \leq b_n = T$ so that

$$\{s \in [t_0, T] \mid y(s) \in \Upsilon_j\} = \bigcup_{k=0}^n [a_k, b_k].$$

This also implies that $\{s \in [t_0, T] \mid y(s) \notin \Upsilon_j\}$ is made of a finite number of open intervals, namely

$$\{s \in [t_0, T] \mid y(s) \notin \Upsilon_j\} = \bigcup_{k=0}^{n-1} (b_k, a_{k+1}).$$

In particular, by the analysis done in Step 1-3 we have that

$$\omega(b_k, y(b_k)) \leq \omega(a_{k+1}, y(a_{k+1})) + \int_{b_k}^{a_{k+1}} \mathcal{L}(y(s), \dot{y}(s)) ds, \quad \forall k \in \{0, \dots, n-1\}. \quad (6.17)$$

We claim that a similar inequality holds true when the trajectory remains on Υ_j , that is

$$\omega(a_k, y(a_k)) \leq \omega(b_k, y(b_k)) + \int_{a_k}^{b_k} \mathcal{L}(y(s), \dot{y}(s)) ds, \quad \forall k \in \{0, \dots, n\}. \quad (6.18)$$

If (6.18) holds true, then combining it with (6.17) we get that

$$\omega(t_0, x_0) \leq \omega(T, y(T)) + \int_{t_0}^T \mathcal{L}(y(s), \dot{y}(s)) ds = \psi(y(T)) + \int_{t_0}^T \mathcal{L}(y(s), \dot{y}(s)) ds = \vartheta(t_0, x_0)$$

Where the last equality comes from the fact that y is an optimal trajectory starting from x_0 at t_0 . Note that this contradicts the assumption $\vartheta(t_0, x_0) < \omega(t_0, x_0)$. Therefore, to complete the proof in the Non-chattering case we only need to show that (6.18) holds.

In this case, the proof runs similarly as the one given in Step 1. We only need to apply Lemma 6.1 with $\mathbb{F}(x) = \cup_{i \in \mathcal{I}_j} f_i(x, \mathcal{A}_{i_j}^0)$ and $\mathbb{L} = \mathcal{L}$. Since each f_i and \mathcal{L}_i are locally Lipschitz, uniformly with respecto to the second variable, we readily check that the following defined a locally Lipschitz set-valued map:

$$x \mapsto \bigcup_{i \in \mathcal{I}_j} \{(v, \ell) \mid v \in f_i(x, \mathcal{A}_i), \ell = \mathcal{L}(x, v)\},$$

and so, the conclusion follows directly from Lemma 6.1.

Chattering case: this situation refers to the circumstance in which the set

$$\{s \in [t_0, T] \mid y(s) \notin \Upsilon_j\}$$

is made of a countably infinite collection of open intervals. To treat this case we use a technique very close to [17, Lemma 3.3].

Let us take $\varepsilon > 0$, we claim that there are:

- $(t_\varepsilon, x_\varepsilon) \in (0, T) \times \mathcal{K}$ with $t_\varepsilon \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and $x_\varepsilon \in \mathcal{B}(x_0, \varepsilon)$,
- $y_\varepsilon \in \mathbb{S}^T(t_\varepsilon, x_\varepsilon)$ that verifies $y_\varepsilon(T) = y(T)$

such that there exists a partition of $[t_\varepsilon, T]$, say $\{t_\varepsilon = \tau_0 < \tau_1 < \dots < \tau_m < \tau_{m+1} = T\}$, so that for any $k \in \{0, \dots, m\}$ we either can find $i \in \mathcal{I}_j$ such that $y_\varepsilon(s) \in \mathcal{M}_i$ all along (τ_k, τ_{k+1}) or $y_\varepsilon(s) \in \Upsilon_j$ all along (τ_k, τ_{k+1}) , and

$$\int_{t_\varepsilon}^T \mathcal{L}(y_\varepsilon(s), \dot{y}_\varepsilon(s)) ds \leq \int_{t_0}^T \mathcal{L}(y(s), \dot{y}(s)) ds + \varepsilon.$$

Indeed, for given $\varepsilon > 0$ we can construct a partition of $[t_0, T]$

$$b_0 := t_0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_n < b_n \leq T =: a_{n+1}$$

that verifies

$$\text{meas} \left(\{s \in [t_0, T] \mid y(s) \notin \Upsilon_j\} \setminus \bigcup_{k=1}^n (a_k, b_k) \right) \leq \varepsilon.$$

with $y(a_k), y(b_k) \in \Upsilon_j$ and $(a_k, b_k) \subseteq \{s \in [t_0, T] \mid y(s) \notin \Upsilon_j\}$ for any $k = 1, \dots, n$. In addition,

$$\bigcup_{k=0}^n [b_k, a_{k+1}] \setminus \{s \in [t_0, T] \mid y(s) \in \Upsilon_j\} = \{s \in [t_0, T] \mid y(s) \notin \Upsilon_j\} \setminus \bigcup_{k=1}^n (a_k, b_k).$$

Hence, if we set $\mathbb{T}_j^k := [b_k, a_{k+1}] \setminus \{s \in [t_0, T] \mid y(s) \in \Upsilon_j\}$ and $\tilde{\varepsilon}_k = \text{meas}(\mathbb{T}_j^k)$, we have $\sum_{k=0}^n \tilde{\varepsilon}_k \leq \varepsilon$.

On the other hand, there must be some $k \in \{0, \dots, n\}$ for which there is a countable family of intervals $(\alpha_p, \beta_p) \subseteq (b_k, a_{k+1})$, pairwise disjoint that verifies

$$\tilde{\varepsilon}_k = \sum_{p \in \mathbb{N}} (\beta_p - \alpha_p), \quad y(t) \in \mathcal{M}_i, \quad \forall t \in (\alpha_p, \beta_p) \text{ for some } i \in \mathcal{I}_j \quad \text{and} \quad y(\alpha_p), y(\beta_p) \in \Upsilon_j.$$

Let $\mathcal{K} \subseteq \{0, \dots, n\}$ be the collection of all indexes for which the latter property holds.

Let $r > 0$ so that $y(s) \in \mathcal{B}(0, r)$ for any $s \in [t_0, T]$. Consider as well $\varepsilon_i > 0$ and $\Delta_i > 0$ the constant given by (H4), and suppose $\varepsilon \leq \varepsilon_i$. So, for any $p \in \mathbb{N}$, if we set $\tau_p = \alpha_p + \Delta_i(\beta_p - \alpha_p)$, we can pick $y_p \in \mathbb{S}^{\tau_p}(\alpha_p, y(\alpha_p))$ and $t_p \in (\alpha_p, \tau_p]$ such that

$$y_p(s) \in \Upsilon_j, \quad \forall s \in [\alpha_p, t_p], \quad y_p(\alpha_p) = y(\alpha_p), \quad \text{and} \quad y_p(t_p) = y(\beta_p).$$

Let $k \in \mathcal{K}$ and consider the measurable function $\mu_k : [b_k, a_{k+1}] \rightarrow \mathbb{R}$ given by

$$\mu_k(s) = \mathbb{1}_{[b_k, a_{k+1}] \setminus \mathbb{T}_j^k}(s) + \sum_{p \in \mathbb{N}} \frac{t_p - \alpha_p}{\beta_p - \alpha_p} \mathbb{1}_{(\alpha_p, \beta_p)}(s) > 0, \quad \forall s \in [b_k, a_{k+1}].$$

Accordingly, the map $s \mapsto \nu_k(s) := b_k + \int_{b_k}^s \mu_k(\tau) d\tau$ defined on $[b_k, a_{k+1}]$ is a homeomorphism from $[b_k, a_{k+1}]$ into some interval $[b_k, c_{k+1}]$, where we have the estimate

$$c_{k+1} - a_{k+1} = \text{meas}([b_k, a_{k+1}] \setminus \mathbb{T}_j^k) - (a_{k+1} - b_k) + \sum_{p \in \mathbb{N}} (t_p - \alpha_p) \leq \Delta_i \tilde{\varepsilon}_k \quad (6.19)$$

using the fact that $(t_p - \alpha_p) \leq \Delta_i(\beta_p - \alpha_p)$.

Consider the measurable function $v_k : [b_k, c_{k+1}] \rightarrow \mathbb{R}^N$ given by

$$v_k(s) = \dot{y}(\nu_k^{-1}(s)) \mathbb{1}_{[b_k, a_{k+1}] \setminus \mathbb{T}_j^k}(\nu_k^{-1}(s)) + \sum_{p \in \mathbb{N}} \dot{y}_p(s) \mathbb{1}_{(\alpha_p, \beta_p)}(\nu_k^{-1}(s)), \quad \text{for a.e. } s \in [b_k, c_{k+1}].$$

Let $y_k : [b_k, c_{k+1}] \rightarrow \mathbb{R}^N$ be defined via

$$y_k(s) = y(b_k) + \int_{b_k}^s v_k(t) dt, \quad \forall s \in [b_l, a_{l+1}].$$

By construction $y_k(\nu_k(t)) = y(t)$ for any $t \in [b_k, a_{k+1}] \setminus \mathbb{T}_j^k$ and $y_k(t) \in \Upsilon_j$ for any $t \in [b_k, c_{k+1}]$. In particular, $y_k(c_{k+1}) = y(a_{k+1})$.

On the other hand, by the Change of Variable Theorem for absolutely continuous function (see for instance [24, Theorem 3.54]) we get

$$\int_{b_k}^{c_{k+1}} \mathcal{L}(y_k(s), \dot{y}_k(s)) ds = \int_{b_k}^{a_{k+1}} \mathcal{L}(y_k(\nu(s)), \dot{y}_k(\nu(s))) \dot{\nu}(s) ds.$$

Furthermore, $\mathcal{L}(y_k(\nu), \dot{y}_k(\nu)) \dot{\nu} = \mathcal{L}(y, u)$ a.e. on $[b_k, a_{k+1}] \setminus \mathbb{T}_j^k$. Note as well that, by the Gronwall Lemma, there is a constant $L > 0$, which depends only on x_0 and T such that

$$\mathcal{L}(y_k(s), \dot{y}_k(s)) \leq L \quad \text{for a.e. } s \in [b_k, a_{k+1}].$$

On the other hand, since $\mathcal{L} \geq 0$ we get

$$\int_{b_k}^{c_{k+1}} \mathcal{L}(y_k(s), \dot{y}_k(s)) ds \leq \int_{b_k}^{a_{k+1}} \mathcal{L}(y(s), \dot{y}(s)) ds + L\tilde{\varepsilon}_k. \quad (6.20)$$

Note that this construction is valid for any $k \in \mathcal{K}$. Else if $k \notin \mathcal{K}$, we just set $c_{k+1} = a_{k+1}$ and $y_k(s) = y(s)$ for any $s \in [b_k, c_{l+1}]$. Therefore, doing the same procedure for each $k \in \{0, \dots, n\}$, we can construct inductively an absolutely continuous curve \tilde{y}_ε in the following way:

- Set first

$$\tilde{y}_\varepsilon(s) = y_0(s), \quad s \in [t_0, t_1], \quad t_1 = c_1.$$

- Then for any $k \in \{1, \dots, n\}$

$$\begin{aligned} \tilde{y}_\varepsilon(s) &= y(a_k - t_{2k-1} + s), & s \in [t_{2k-1}, t_{2k}], & & t_{2k} &= t_{2k-1} + b_k - a_k \\ \tilde{y}_\varepsilon(s) &= y_k(b_k - t_{2k} + s), & s \in [t_{2k}, t_{2k+1}], & & t_{2k+1} &= t_{2k} + c_{k+1} - b_k. \end{aligned}$$

- Finally, $\tilde{y}_\varepsilon(s) = y(a_{n+1} - t_{2n+1} + s)$ for $s \in [t_{2n+1}, T_\varepsilon]$ with $T_\varepsilon = t_{2n+1} + T - a_{n+1}$.

Notice that $c_{k+1} - b_k \geq \text{meas}([b_k, a_{k+1}] \cap \mathbb{T}_j^k) = a_{k+1} - b_k - \tilde{\varepsilon}_k$. Hence, after a few algebraic steps we obtain, by virtue of (6.19),

$$T_\varepsilon = T + \sum_{l=0}^n (c_{l+1} - a_{l+1}) \in [T - \varepsilon, T + \Delta_i \varepsilon].$$

Moreover

$$\int_{t_0}^{T_\varepsilon} \mathcal{L}(\tilde{y}_\varepsilon(s), \dot{\tilde{y}}_\varepsilon(s)) ds \leq \int_{t_0}^T \mathcal{L}(y, \dot{y}) ds + L \sum_{k=0}^n \tilde{\varepsilon}_k \leq \int_{t_0}^T \mathcal{L}(y, \dot{y}) ds + L\varepsilon$$

To summarize, we have constructed a trajectory of the control systems and a new horizon time $T_\varepsilon > 0$ for which the sets $\{s \in [t, T_\varepsilon] \mid y_\varepsilon(s) \in \Upsilon_j\}$ and $\{s \in [t, T_\varepsilon] \mid y_\varepsilon(s) \notin \Upsilon_j\}$ can be decomposed into a finite number of intervals. Furthermore, this trajectory verifies $\tilde{y}_\varepsilon(t) = x$ and $\tilde{y}_\varepsilon(T_\varepsilon) = y(T)$.

Since $\varepsilon > 0$ is arbitrary and $\Delta_i > 0$ does not depends upon ε , we may assume that $T_\varepsilon \in (T - \varepsilon, T + \varepsilon)$; using $\min\left\{\varepsilon, \frac{1}{\Delta_i} \varepsilon, \frac{\varepsilon}{L}\right\}$ instead of ε for instance. Finally, re-scaling ε if necessary, we can assume that

$$|\tilde{y}_\varepsilon(t_0 + T_\varepsilon - T) - x_0| \leq \varepsilon.$$

Therefore, the proof of the claim follows by taking

- $t_\varepsilon = t_0 - T_\varepsilon + T$ and $x_\varepsilon = x_0$ if $T_\varepsilon \leq T$,
- otherwise we set $t_\varepsilon = t_0$ and $x_\varepsilon = \tilde{y}_\varepsilon(t_0 + T_\varepsilon - T)$.

In any case, we set $y_\varepsilon(s) = \tilde{y}_\varepsilon(s - T + T_\varepsilon)$ for any $s \in [t_\varepsilon, T]$, and then we have that

$$\int_{t_\varepsilon}^T \mathcal{L}(y_\varepsilon(s), \dot{y}_\varepsilon(s)) ds \leq \int_{t_0}^T \mathcal{L}(y, \dot{y}) ds + \varepsilon.$$

We now apply the arguments of the Non-Chattering case to the trajectory y_ε and we obtain

$$\omega(t_\varepsilon, x_\varepsilon) \leq \omega(T, y_\varepsilon(T)) + \int_{t_\varepsilon}^T \mathcal{L}(y_\varepsilon(s), \dot{y}_\varepsilon(s)) ds = \psi(y_\varepsilon(T)) + \int_{t_0}^T \mathcal{L}(y_\varepsilon(s), \dot{y}_\varepsilon(s)) ds = \vartheta(t_0, x_0) + \varepsilon.$$

Finally, $\varepsilon > 0$ being arbitrary and ω being lower semicontinuous yield, in the light of the preceding inequality, to $\omega(t_0, x_0) \leq \vartheta(t_0, x_0)$, which contradicts the initial assumption, and thus, the proof of Theorem 3.2 is now complete. □

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