



# Payload optimization for multi-stage launchers using HJB approach and application to a SSO mission

Olivier Bokanowski, Eric Bourgeois, Anna Desilles, Hasnaa Zidani

## ► To cite this version:

Olivier Bokanowski, Eric Bourgeois, Anna Desilles, Hasnaa Zidani. Payload optimization for multi-stage launchers using HJB approach and application to a SSO mission. The 20th IFAC World Congress, Jul 2017, Toulouse, France. 2017, <10.1016/j.ifacol.2017.08.647>. <hal-01598665>

**HAL Id: hal-01598665**

**<https://hal-ensta.archives-ouvertes.fr/hal-01598665>**

Submitted on 29 Sep 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Payload optimization for multi-stage launchers using HJB approach and application to a SSO mission <sup>\*</sup>

Olivier Bokanowski <sup>\*</sup> Eric Bourgeois <sup>\*\*</sup>  
Anna Désilles, Hasnaa Zidani <sup>\*\*\*</sup>

<sup>\*</sup> *Laboratoire Jacques-Louis Lions, Université Paris-Diderot, 5 Rue Thomas Mann, 75205 Paris Cedex13, France, and Laboratoire UMA, Ensta ParisTech, Palaiseau (boka@math.univ-paris-diderot.fr)*

<sup>\*\*</sup> *CNES Launcher Directorate, 52 rue Jacques Hillairet, 75012 Paris, France (eric.bourgeois@cnes.fr)*

<sup>\*\*\*</sup> *Unité des Mathématiques Appliquées (UMA), Ensta ParisTech, 828 Bd des Maréchaux, 91762 Palaiseau Cedex, France*

*(anna.desilles@ensta-paristech.fr, hasnaa.zidani@ensta-paristech.fr)*

---

**Abstract:** This paper deals with a payload optimization problem for three-stage space launcher. The mission of the launch vehicle is to put the payload on a sun-synchronous (SSO) orbit. The considered flight sequence includes two boosts. The first one steers the launcher to a transfer orbit. Then, after a ballistic flight, a second boost is used to perform the orbit transfer manoeuvre to inject the payload to the targeted SSO orbit. The optimization method presented here is based on the Hamilton-Jacobi-Bellman (HJB) approach for hybrid dynamical systems.

*Keywords:* optimal trajectory, optimal control problem, Hybrid systems, Hamilton-Jacobi approach, reachability analysis.

---

## 1. INTRODUCTION

This work concerns the design of a global trajectory optimization procedure for a space shuttle of type Ariane 5, with the aim of steering the maximal payload from Earth to a given SSO orbit. Trajectory optimization for aerospace launchers has been extensively studied in the literature, see for instance Betts (2001); Bonnans et al. (2008); Oberle (1990); Zondervan et al. (1984); Bérend et al. (2005); Ponsard et al. (2009); Bokanowski et al. (2015); Bourgeois et al. (2015) and the references therein.

In the present work, the propellant consumption during the flight is fixed, defined by the rocket design. The goal is to maximize the payload to be launched on a SSO orbit. It is assumed that the upper stage performs two boosts. A first boost steers the launcher to a given GTO target. Then, after a ballistic phase, a second boost is used to perform the transfer manoeuvre from a GTO to the SSO. The optimization problem aims at maximizing the payload mass on the launcher. More precisely, the complete flight sequence is composed of 4 important phases (atmospheric phase, propulsion with first stage until exhaustion of the ergol, propulsion with the first stage until injection on a GTO, ballistic flight until injection on the SSO).

Our approach is based on the reformulation of the trajectory optimization problem as a reachability problem that we solve by using an efficient method based on Hamilton-Jacobi-Bellman (HJB) approach in optimal control theory, see Bokanowski et al. (2010). A similar approach has been first introduced by the authors in Bokanowski et al. (2015); Bourgeois et al. (2015) to solve the propellant optimization problem to steer a given payload to the geostationary orbit (GEO). In that context the orbital transfer manoeuvre from the GTO to the GEO was assumed to be of impulsive type. This hypothesis allows to estimate the amount of propellant needed for the orbital transfer using Tseolkovsky's formula. Here the problem is considered in a more complete setting where a ballistic flight phase is considered followed by a second boost for the orbital transfer. The duration of each boost is not fixed and is considered as an optimization parameter. The problem involves also the shooting azimuth and inclination speed as optimization parameters. In addition to these parameters, the launcher is controlled by two functions; namely, the incidence and sideslip angles (time-dependent functions), that are considered as control functions. The mathematical formulation of the problem leads to a control optimal problem for a hybrid system.

This complex non-linear problem presents several challenges for the implementation of HJB approach. In Bokanowski et al. (2015), the authors had developed an approach to combine the HJB framework with parameter optimization methods without increasing the state space dimension. It is shown here how the method can be applied to the more general two-boost problem in the context of

---

<sup>\*</sup> This work is partially supported by Centre National d'Études Spatiales (CNES), under the grant RT-NT-4310000-ZZ-1602-ENST, by the French National Research Agency (ANR) through the "iCODE Institute project" funded by the IDEX Paris-Saclay ANR-11-IDEX-0003-02, and by the DGA research grant 0660037

a SSO mission. The proposed optimization procedure is supported by theoretical results that show the convergence of the overall procedure to a global optimal solution.

## 2. POSITION OF THE PROBLEM

### 2.1 The mission

This problem concerns a trajectory optimization problem for a multi-stage heavy launch vehicle.

**Launch vehicle.** An Ariane 5 like launch vehicle is considered here. A description of the characteristics of the launcher (propellant, structure masses, etc.) can be found in Bokanowski et al. (2015).

**Target orbit :** a circular sun-synchronous orbit (SSO) with the following parameters:

- inclination :  $i = 98.6$  degrees;
- eccentricity  $e = 0$ ;
- altitude :  $r = 800$  km;
- longitude of the ascending node : free.

**The launch site** is Kourou, French Guyana.

### 2.2 Flight sequence

Let  $t \geq 0$  represent the flight time variable. In this work the flight sequence is decomposed into the following phases:

**Phase 0 (atmospheric phase).** The flight starts at  $t = 0$  when the vehicle leaves the launch base. Both boosters along with the stage  $E_1$  are ignited and consume propellant with flow rates  $\beta_{EAP}$  and  $\beta_{E1}(t)$  respectively. In this phase, the control law of the launcher is entirely defined by two constant parameters (that will have to be optimized) : the shooting azimuth  $\psi$ , and the inclination speed  $\omega$ . The phase ends at time  $t = t_0$  when the boosters are ejected, once all the propellant is consumed. In the present work  $t_0 = 165$  seconds. Let  $X_0$  be the set of all possible states of the launcher (position and velocity) that can be reached at time  $t_0$  corresponding to a large sample (denoted  $P_{\text{Ini}}$ ) of the parameters  $(\psi, \omega)$ . This set can be obtained by a simple integration of the motion's equations. Each point of  $X_0$  is the image by a known application  $\Gamma$  of some shooting parameters  $(\psi, \omega)$ :

$$x \in X_0 \Leftrightarrow \exists p = (\psi, \omega) \in P_{\text{Ini}}, x = \Gamma(p).$$

Let  $M_{E,i}$ ,  $i = 1, 2$  be the constant net masses of the first and second stages,  $m_F$ , the mass of the fairing and  $M_{P,i}$ ,  $i = 1, 2$  the masses of the propellants of first and second stages respectively. The propellant consumption flow  $\beta_{E1}(\tau)$  is a known function of time. Hence the mass of the launcher *without the payload*, at time  $t_0$ , denoted  $M_0^{S,P}$ , is:

$$M_0^{S,P} := \underbrace{\sum_{i=1,2} M_{E,i} + m_F}_{\text{Structure}} + \underbrace{\sum_{i=1,2} M_{P,i} - \int_0^{t_0} \beta_{E1}(\tau) d\tau}_{\text{Propellant}}. \quad (1)$$

*Remark 1.* At the end of this phase the launcher is assumed to be in the thermospheric region. So, for all the following phases we will assume that the (instantaneous) aerodynamic forces can be neglected.

**Phase 1.** This phase starts at time  $t_0$  and ends at time  $t_1$ . The fairing is ejected during this phase at a given time  $t_0 < t_{fair} < t_1$ . The thrust force is provided by the first stage engine only. The ending time  $t_1$  of this phase is known and corresponds to the entire consumption of the propellant of the first stage. During this phase the mass of the launcher (without the payload) is known and given by:

$$M^{S,P}(t) := \sum_{i=1,2} M_{E,i} + M_{P,1}(t) + M_{P,2} + \begin{cases} m_F, & \text{if } t < t_{fair} \\ 0, & \text{if } t \geq t_{fair} \end{cases} \quad (2)$$

where  $M_{P,1}(t)$  is deduced from the knowledge of  $\beta_{E1}$  :

$$M_{P,1}(t) := M_{P,1} - \int_0^t \beta_{E1}(\tau) d\tau, \quad t \in [t_0, t_1]. \quad (3)$$

**Phase 2.** This phase starts at time  $t_1$  with ignition of the second stage engine. The ending time  $t_2$  corresponding to the extinction of the engine of the second stage is unknown and have to be optimized. During this phase the mass of the launcher (payload excluded) is defined explicitly by:

$$M^{S,P}(t) := M_{E,2} + M_{P,2} - \beta_{E2}(t - t_1), \quad t \in [t_1, t_2], \quad (4)$$

where  $\beta_{E2}$  is the propellant first engine flux (a constant).

**Phase 3: the ballistic flight.** The duration of this phase is unknown and has to be optimized  $\tau_B \geq 0$  ( $\tau_B$  may be 0 if a ballistic flight is not necessary to reach the target with an optimal payload mass). All engines are off during this phase of the flight and the mass of the launcher remains constant :

$$M^{S,P}(t) = M^{S,P}(t_2), \quad t \in [t_2, t_3] \quad (5)$$

with  $t_3 := t_2 + 2 + \tau_B$ . The orbital characteristics of the transfer orbit are also unknown and have to be optimized.

**Phase 4.** This flight phase starts at time  $t_3$ , when the engine of the second stage are ignited again. This phase lasts until the total consumption of the propellant of the second stage, until some unknown final time denoted  $t_f$ .

### 2.3 Optimization problem

The considered problem is to determine :

- the shooting parameters  $(\psi, \omega)$ ,
- the duration of the first boost of the second engine  $t_2 - t_1$
- the duration of the ballistic flight  $\tau_B = t_3 - t_2$
- the control laws of the phases 1, 2 and 4

in order to

*maximize the payload mass  $m$  on the SSO orbit.*

In particular all the propellant mass has to be consumed at the end of the mission.

### 2.4 Motion equations

By using standard arguments (see Bokanowski et al. (2015) for instance), the state of the launch vehicle can be represented in the spherical coordinates  $(r, L, \ell)$  for the position vector and  $(v, \gamma, \chi)$  for the velocity vector. The advantage of this coordinate system is that it is possible to isolate the differential equation of the longitude variable

$L$  because the motion equations of the other variables do not depend on it. Let

$$M(t) := m + M^{S,P}(t)$$

be the total mass of the launcher, composed of the payload mass  $m > 0$ , and of the structure and propellant mass  $M^{S,P}(t)$ . As mentioned above, the aerodynamic forces are neglected after Phase 0 and therefore during the phases 1, 2 and 4, the launcher is subject to the following forces.

**Gravitational force:** it is defined by  $\vec{F}_g = M\vec{g}$  where  $\vec{g}$  is the gravitational field.

**Thrust force:** it is assumed that the direction of the thrust force coincides with the axis of the launcher. Its orientation is defined by the incidence angle  $\alpha(t)$  and the sideslip angle  $\delta(t)$ , which are both function of time and will be considered as control functions. The modulus of the thrust force is given by  $F_T(r) := \beta g_0 I_{sp} - SP(r)$  where  $g_0 := 9.81 \text{ ms}^{-2}$ ,  $P(r)$  is the atmospheric pressure,  $\beta$  is the propellant flow rate,  $I_{sp}$  is the specific impulse and  $S$  is a surface coefficient.

**Coriolis force  $\vec{F}_{CO}$  and centripetal force  $\vec{F}_{CP}$ :**

$$\vec{F}_{CO} = 2M\vec{\Omega} \wedge \vec{V} \quad \text{and} \quad \vec{F}_{CP} = M\vec{\Omega} \wedge (\vec{\Omega} \wedge \vec{OG}),$$

where  $\vec{\Omega}$  is the Earth's angular velocity. These two forces are taken into account because the launchers's trajectory is represented in a relative reference frame and not in an inertial one.

By using the fundamental law of the dynamics and taking into account the previous forces, the motion of the launcher during the flight phases 2, 3 and 4 is described by the following system of differential equations:

$$\begin{aligned} \frac{dr}{dt} &= v \sin \gamma \\ \frac{d\ell}{dt} &= \frac{v}{r} \cos \gamma \cos \chi \\ \frac{dv}{dt} &= -g_r \sin \gamma + g_\ell \cos \gamma \cos \chi + \frac{F_T \cos \alpha \cos \delta}{M(t)} + F_v^C \\ \frac{d\gamma}{dt} &= \cos \gamma \left( \frac{v}{r} - \frac{g_r}{v} \right) - \sin \gamma \cos \chi \frac{g_\ell}{v} - \frac{F_T \sin \alpha}{M(t)v} + F_\gamma^C \\ \frac{d\chi}{dt} &= -\frac{g_\ell \sin \chi}{v \cos \gamma} - \frac{v \cos \gamma \tan \ell \sin \chi}{r} + \frac{F_T \cos \alpha \sin \delta}{M(t)v \cos \gamma} + F_\chi^C \end{aligned} \quad (6)$$

where  $g_r$  and  $g_\ell$  are components of the gravitational field including  $J_2$  second order corrections:

$$\begin{aligned} g_r &:= \frac{\mu}{r^2} \left( 1 + J_2 \left( \frac{r_T}{r} \right)^2 (1 - 3 \sin^2 \ell) \right) \\ g_\ell &:= -2 \frac{\mu}{r^2} J_2 \left( \frac{r_T}{r} \right)^2 \sin \ell \cos \ell, \end{aligned}$$

$r_T$  is the earth mean radius,  $\mu$  is the Earth's gravitational constant, and

$$\begin{aligned} F_v^C &:= \Omega^2 r \cos \ell (\sin \gamma \cos \ell - \cos \gamma \sin \ell \cos \chi) \\ F_\gamma^C &:= 2\Omega \cos \ell \sin \chi + \frac{\Omega^2 r}{v} \cos \ell (\cos \gamma \cos \ell + \sin \gamma \sin \ell \cos \chi) \\ F_\chi^C &:= \frac{\Omega^2 r \sin \ell \cos \ell \sin \chi}{v \cos \gamma} - 2\Omega (\sin \ell - \tan \gamma \cos \ell \cos \chi) \end{aligned}$$

### 3. MATHEMATICAL FORMULATION. OPTIMAL CONTROL PROBLEM OF A HYBRID SYSTEM

#### 3.1 Introduction of the "consumption time" variable

The final time  $t_f$  of the mission is unknown because it depends on the duration of the ballistic flight,  $\tau_B$  that has to be optimized. As all the propellant of the launcher must be consumed at the end of the mission, the sum of the durations of the phases 1 and 4 of the flight corresponds to the entire consumption of the second  $E_2$  engine, leading to:  $(t_2 - t_1) + (t_f - t_3) = \frac{M_{P,2}}{\beta_{E2}}$ . Then, the total consumption time is given by:

$$T := \frac{M_{P,E2}}{\beta_{E2}} + t_1 - t_0. \quad (7)$$

The idea now is to introduce a new time variable  $s$  that

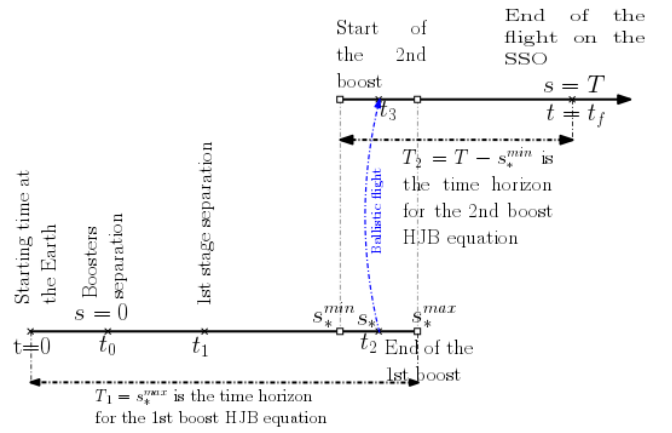


Fig. 1. Relations between the physical time variable  $t$  and the "consumption" variable  $s$

corresponds to the duration of the propellant consumption during the phases 1, 2 and 4 of the flight, as follows:

$$s := \begin{cases} t - t_0 & \text{if } t \in [t_0, t_2[ \\ t_2 - t_0 & \text{if } t \in [t_2, t_3[ \\ t - t_3 + (t_2 - t_0) & \text{if } t \in [t_3, t_f] \end{cases} \quad (8)$$

and we will denote  $s = s(t)$  the above correspondence. It is clear that  $s \in [0, T]$  and that  $s = 0$  for  $t = t_0$ . Observe also that  $s(\cdot)$  is continuous, and remains constant equal to  $s_* = s(t_2) = t_2 - t_0$  for  $t \in [t_2, t_3]$ . Conversely, we can define  $t = t(s)$ , then  $t(s)$  will have a jump at  $s_* = t_2 - t_0$ :  $t(s_*^-) = t_2$ , and  $t(s_*^+) = t_3$ . In the sequel we will denote  $s_* := t_2 - t_1$  the duration of the first boost of the second stage engine. This value is an optimization variable. Figure 1 illustrates the definition of the consumption time variable  $s$ .

#### 3.2 Dynamical system and state transfer function

Let  $x = (r, \ell, v, \gamma, \chi) \in \mathbb{R}^5$  be the physical state vector and let  $m \in \mathbb{R}_+$  be the payload mass. This mass is constant during the flight, hence its evolution equation is simply

$$\dot{m} = 0. \quad (9)$$

Let us denote

$$y := (x, m) = (r, \ell, v, \gamma, \chi, m) \in \mathbb{R}^6 \quad (10)$$

the state vector of the system.

First, we introduce the transfer function for the ballistic flight phase. During this phase, the trust force is zero ( $F_T = 0$ ), because all engines are off, and the right hand side of the equations (6) does not depend on the mass  $M$  anymore. Therefore, the launcher's motion is governed by an uncontrolled and autonomous differential system:

$$\begin{cases} \mathbf{z}'(t) = \varphi(\mathbf{z}(t)), & t \in [t_2, t_3] \\ \mathbf{z}(t_2) = y_0 \end{cases} \quad (11)$$

where  $\varphi(\mathbf{z})$  denotes the function of the right hand side of (13) when  $F_T = 0$  (no thrust force). Now, introduce the associated flow map  $\Phi(t, y_0)$ , so that the solution of (11) satisfies  $\mathbf{z}(t) = \Phi(t, y_0)$ . The application  $\Phi$  can be considered as a state transfer function that associates to a terminal state of the phase 2 of the flight  $\mathbf{z}(t_2) = \mathbf{y}(s(t_2)^-) = \mathbf{y}(s_*^-)$  the starting state of phase 3  $\mathbf{z}(t_3) = \mathbf{y}(s(t_3)^+) = \mathbf{y}(s_*^+)$ , by

$$\mathbf{y}(s_*^+) = \Phi(\tau_B, \mathbf{y}(s_*^-)), \quad (12)$$

where  $\tau_B$  is the duration of the ballistic flight (see the description of Phase 3). Now, during phases 1, 2 and 4 (i.e., for  $t \notin [t_2, t_3]$ ), the system is controlled by  $\mathbf{u} = (\alpha, \delta)$ , the two orientation angles of the trust force. By using the time variable transformation (8) and the fact that  $M(t) = M(t(s))$ , we can write all time-dependent quantities of equations (6) as functions of the variable  $s$ . Therefore the system of differential equations (6) and (9) can be re-written as:

$$\begin{cases} \mathbf{y}'(s) = f(s, \mathbf{y}(s), \mathbf{u}(s)), & s \in [0, s_*[ \\ \mathbf{y}(s_*^+) = \Phi(\tau_B, \mathbf{y}(s_*^-)) \\ \mathbf{y}'(s) = f(s, \mathbf{y}(s), \mathbf{u}(s)), & s \in ]s_*, T] \end{cases} \quad (13)$$

where the function  $f$  represents the right hand sides of (6) and (9), and  $s$  is the consumption time variable.

### 3.3 State constraints

Because the target orbit is at a low altitude, a special constraint on the dynamic thermal flow has to be satisfied during the phase 2 of the flight:

$$0.5 \rho(r) v^3 \leq 555 \text{ Wm}^{-2} \quad (14)$$

where  $\rho(r)$  is the density of the atmosphere at altitude  $r$ . The model of density for atmosphere at high altitude used in this work is defined by a tabulated function provided by the CNES<sup>1</sup>. Then the set of state constraints, in  $\mathbb{R}^6$ , is defined by:

$$\mathcal{K} := \left\{ y = (x, m) \in \mathbb{R}^6, \quad 0.5 \rho(r) v^3 \leq 555 \right\}. \quad (15)$$

### 3.4 Target set

The target set is the SSO circular orbit defined in section 2.1. In order to represent it in the space of spherical coordinates  $(r, L, \ell, v, \gamma, \chi) \in \mathbb{R}^6$ , standard formulas from orbital mechanics are used. In particular one can express the eccentricity  $e(x)$ , the major semi-axis  $a(x)$  and the inclination  $i(x)$  as functions of the five spherical coordinates  $x \in \mathbb{R}^5$  (see appendix for more details). Then the target set, in  $\mathbb{R}^6$ , is defined by:

$$\mathcal{C} := \left\{ y = (x, m) \in \mathbb{R}^6, \text{ s.t. } e(x) = 0, a(x) = 800, \right. \\ \left. i(x) = 98.6^\circ, m \geq 0 \right\}. \quad (16)$$

### 3.5 Optimal control problem

Denote  $\mathbf{y}_y^{\mathbf{u}}(\cdot) = (\mathbf{x}_y^{\mathbf{u}}(\cdot), \mathbf{m}_y^{\mathbf{u}}(\cdot))$  the solution of (13), with initial data  $\mathbf{y}_y^{\mathbf{u}}(0) = y$ , and for controls  $\mathbf{u}(\cdot)$  in a the set of admissible controls  $\mathcal{U}_{ad}$ :

$$\mathcal{U}_{ad} := \{ \mathbf{u} : (0, T) \rightarrow \mathbb{R}^2 \text{ measurable, } \mathbf{u}(s) \in U \text{ a.e.} \}.$$

with

$$U := [0, 2\pi] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \quad (17)$$

The optimal control problem ( $\mathcal{P}$ ) corresponding to the SSO mission can be formulated as follows:

$$\begin{cases} \sup \mathbf{m}_y^{\mathbf{u}}(T) \\ \text{s. t.} \\ (i) \quad \mathbf{y}_y^{\mathbf{u}} = (\mathbf{x}_y^{\mathbf{u}}, \mathbf{m}_y^{\mathbf{u}}) \text{ is solution of (13) associated to } \mathbf{u} \\ \quad \text{with } \mathbf{y}(0) = y \in X_0 \times [0, +\infty[, \\ (ii) \quad \mathbf{y}_y^{\mathbf{u}}(s) \in \mathcal{K}, \quad \forall s \in [0, T], \\ (iii) \quad s_* \in [s_2^{\min}, s_2^{\max}] \text{ and } \tau_B \in [0, \tau_B^{\max}] \\ \quad \text{with } \mathbf{y}(s_*^+) = \Phi(\tau_B, \mathbf{y}(s_*^-)) \\ (iv) \quad \mathbf{y}_y^{\mathbf{u}}(T) \in \mathcal{C}. \end{cases} \quad (\mathcal{P})$$

As mentioned earlier, the duration of the ballistic flight  $\tau_B$  may be equal to 0. The control problem can be seen as a control of a hybrid system where the control strategy includes (at most) one possible switch. The switching is controlled by  $\tau_B \in [0, \tau_B^{\max}]$ , the duration of the ballistic flight. The value of the bounds  $s_2^{\min}$ ,  $s_2^{\max}$  and  $\tau_B^{\max}$  can be chosen by using some physical considerations. For example, one can consider  $s_2^{\min} = t_1 - t_0$ ,  $s_2^{\max} = T$  and  $\tau_B^{\max} = T$ .

## 4. HJB APPROACH

The method that will be used to solve the problem ( $\mathcal{P}$ ) is based on the definition of two reachability problems associated with different phases of the flight. Here, some notations are needed to define the problem. In all the sequel, for any subset  $\mathcal{S}$  of  $\mathbb{R}^6$  with a boundary  $\partial\mathcal{S}$ ,  $d_{\mathcal{S}}$  denotes the signed distance function to  $\mathcal{S}$ , defined by  $d_{\mathcal{S}}(y) = \text{dist}(y, \partial\mathcal{S})$  if  $y \in \mathcal{S}$ , and  $d_{\mathcal{S}}(y) = -\text{dist}(y, \partial\mathcal{S})$  otherwise.

### 4.1 A reachability problem associated to the second boost of the second stage (phase 4).

From now on, for any  $s \in [0, T]$  and  $y \in \mathbb{R}^6$ , we denote by  $\mathbf{y}_{s,y}^{\mathbf{u}}(\cdot) = (\mathbf{x}_{s,y}^{\mathbf{u}}(\cdot), \mathbf{m}_{s,y}^{\mathbf{u}}(\cdot))$  the trajectory starting in  $y$  at the initial time  $s$  and associated to  $\mathbf{u}(\cdot) \in \mathcal{U}_{ad}$ :

$$\begin{cases} \mathbf{y}'(\xi) = f(\xi, \mathbf{y}(\xi), \mathbf{u}(\xi)), & \xi \in [0, s_*[ \\ \mathbf{y}(s_*^+) = \Phi(\tau_B, \mathbf{y}(s_*^-)) \\ \mathbf{y}'(\xi) = f(\xi, \mathbf{y}(\xi), \mathbf{u}(\xi)), & \xi \in ]s_*, T] \\ \mathbf{y}(s) = y \end{cases} \quad (18)$$

<sup>1</sup> Centre National d'Études Spatiales

Let us define the value function, for  $s \in [s_*^{min}, T]$  and  $y \in \mathbb{R}^6$ , by:

$$w_0(s, y) = \inf_{\mathbf{u} \in \mathcal{U}_{ad}} \left\{ d_{\mathcal{C}}(\mathbf{y}_{s,y}^{\mathbf{u}}(T)) \bigvee \max_{\xi \in [s, T]} d_{\mathcal{K}}(\mathbf{y}_{s,y}^{\mathbf{u}}(\xi)) \right\} \quad (19)$$

(where  $a \vee b := \max(a, b)$ ).

The control problem (19) corresponds to a classical maximum running cost problem with fixed horizon  $T$ . Following Bokanowski et al. (2010); Altarovici et al. (2013), the following result holds.

*Theorem 2.* (i) The function  $w_0$  is the unique Lipschitz continuous viscosity solution of the following HJB equation on  $[s_*^{min}, T] \times \mathbb{R}^6$ :

$$\min(-\partial_s w_0(s, y) + H(s, y, D_y w_0(s, y)), w_0(s, y) - d_{\mathcal{K}}(y)) = 0, \\ w_0(T, y) = d_{\mathcal{C}}(y) \bigvee d_{\mathcal{K}}(y),$$

where for any  $s \in [0, T]$ ,  $y \in \mathcal{K}$  and  $q \in \mathbb{R}^6$ :

$$H(s, y, q) := \max_{u \in U} -\langle f(s, y, u), q \rangle$$

(and where  $\partial_s w$  and  $D_y w$  represent respectively the time and space derivatives, and  $U$  is defined as in (17)).

(ii) Moreover, we have:

$$w_0(s, y) \leq 0 \Leftrightarrow \forall \varepsilon > 0, \exists \mathbf{u}_\varepsilon \in \mathcal{U}_{ad}, d_{\mathcal{C}}(\mathbf{y}_{s,y}^{\mathbf{u}_\varepsilon}(T)) \leq \varepsilon \\ \text{and } d_{\mathcal{K}}(\mathbf{y}_{s,y}^{\mathbf{u}_\varepsilon}(\xi)) \leq \varepsilon \forall \xi \in [s, T]. \quad (21)$$

For a detailed definition of the viscosity notion, we refer to the monograph of Bardi and Capuzzo-Dolcetta (1997). Assertion (ii) amounts to saying that if  $w_0(s, y) \leq 0$  then there exists a trajectory  $\mathbf{y}_{s,y}$  that is as close as desired to the target  $\mathcal{C}$ .

#### 4.2 A reachability problem associated to problem (P)

Now, define a more general control problem, for  $s \in [0, T]$  and  $y \in \mathbb{R}^6$ , by:

$$w(s, y) = \inf_{\substack{\mathbf{u} \in \mathcal{U}_{ad}, \\ \tau_B \in [0, \tau_B^{max}], \\ s_* \in [s_*^{min}, s_*^{max}]}} \left\{ d_{\mathcal{C}}(\mathbf{y}_{s,y}^{\mathbf{u}}(T)) \bigvee \max_{\xi \in [s, T]} d_{\mathcal{K}}(\mathbf{y}_{s,y}^{\mathbf{u}}(\xi)) \right\}. \quad (22)$$

To characterize the value function  $w$  as a solution of HJB equation, we need to introduce the "jump" operator  $\mathcal{M}$  defined, for any Lipschitz continuous function  $\sigma$ , by:

$$\mathcal{M}\sigma(s, y) := \min_{\tau \in [0, \tau_B^{max}]} \sigma(s, \Phi(\tau, x)).$$

By using the viscosity notion in HJB theory, one can prove the following result.

*Theorem 3.* (i) The function  $w$  is Lipschitz continuous. (ii) We have:

$$w(s, y) = w_0(s, y) \quad \forall s \in [s_*^{max}, T], \forall y \in \mathbb{R}^6.$$

(iii) The function  $w$  is the unique Lipschitz continuous viscosity solution of the following HJB equation:

$$\min \left( -\partial_s w + H(s, y, D_y w), w - d_{\mathcal{K}}(y) \right) = 0, \\ \text{on } (s_*^{max}, T) \times \mathbb{R}^6, \quad (23a)$$

$$\min \left( -\partial_s w + H(s, y, D_y w), w - d_{\mathcal{K}}(y) \right. \\ \left. w - \mathcal{M}w_0(s, y) \right) = 0 \quad \text{on } (s_*^{min}, s_*^{max}) \times \mathbb{R}^6 \quad (23b)$$

$$\min \left( -\partial_s w + H(s, y, D_y w), w - d_{\mathcal{C}}(y) \right) = 0, \\ \text{on } (0, s_*^{min}) \times \mathbb{R}^6, \quad (23c)$$

$$w(T, y) = d_{\mathcal{C}}(y) \bigvee d_{\mathcal{K}}(y) \quad \text{for } y \in \mathbb{R}^6. \quad (23d)$$

Equation (23a) together with the final boundary condition (23d) confirm the assertion that  $w \equiv w_0$  on  $[s_*^{max}, T] \times \mathbb{R}^d$ . Also, equation (23b) indicates that the optimal strategy may include at most one switch at time  $s_* \in [s_*^{min}, s_*^{max}]$ .

By definition of the value function  $w$ , it follows that:

$$w(0, y) \leq 0 \Leftrightarrow \exists s_* \in [s_*^{min}, s_*^{max}], \exists \tau_B \in [0, \tau_B^{max}] \\ \exists \mathbf{u}_1 \text{ on } [0, s_*] \text{ s.t. } w_0(s_*, \Phi(\tau_B, \mathbf{y}_{s_*}^{\mathbf{u}_1}(s_*))) \leq 0 \quad (24)$$

#### 4.3 Resolution procedure for the problem (P)

Let  $y = (x(p), m) \in X_0 \times [0, +\infty[$  be an initial state composed of the physical state  $x(p) \in X_0$ , state of the launcher at the end of Phase 0 of the flight corresponding to a choice of shooting parameters  $p \in P_{ini}$  and payload mass  $m$ . From (21) and (24), it follows that if  $w(0, y) \leq 0$ , then there exists a time  $s_* \in [s_*^{min}, s_*^{max}]$ , a ballistic time  $\tau_B \in [0, \tau_B^{max}]$ , and a control law  $\mathbf{u}(\cdot)$  defined on  $[0, T]$  of the form:

$$\mathbf{u}(s) = \begin{cases} \mathbf{u}_1(s) & \text{if } s \in [0, s_*[ \\ \mathbf{u}_2(s) & \text{if } s \in [s_*, T[ \end{cases} \quad (25)$$

such that the corresponding solution  $\mathbf{y}_y^{\mathbf{u}}$  satisfies :

$$\begin{cases} \mathbf{y}(0) = y \in X_0 \times [0, +\infty[, \mathbf{y}_y^{\mathbf{u}}(s) \in \mathcal{K} \forall s \in [0, T] \\ \mathbf{y}(s_*^+) = \Phi(\tau_B, \mathbf{y}(s_*^-)), \text{ and } \mathbf{y}_y^{\mathbf{u}}(T) \in \mathcal{C}. \end{cases}$$

Therefore, to solve the problem (P), one can proceed as follows:

- *STEP 1.* Compute the set  $X_0$  of possible states of the launcher that can be reached at the end of Phase 0 for a large sample of parameters  $(\psi, \omega)$ .

- *STEP 2.* Solve the HJB equation of Theorem 2 to get an approximation of  $w_0$ .

- *STEP 3.* Solve the HJB equation (23) to obtain an approximation of the value function  $w$ .

- *STEP 4.* Define on the set  $X_0$  the function

$$m^*(x) = \sup\{m \mid w(0, (x, m)) \leq 0\}. \quad (26)$$

This function corresponds to the biggest payload mass that is possible to steer to the SSO starting from  $x$ . Finally, the *optimal mass* is given by:

$$m_{opt} = \sup_{x \in X_0} m^*(x).$$

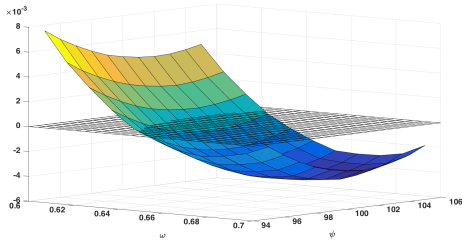


Fig. 2. Values of  $w(0, (x, m^*(x)))$  with  $x = \Gamma(p)$ , for different shooting parameters  $p = (\psi, \omega)$ .

- *STEP 5: Reconstruction of an optimal trajectory.* Let  $x^* \in X_0$  be such that  $m^*(x^*) = m_{opt}$ . By the definition of the set  $X_0$  one can identify the optimal shooting parameters  $p^* \in P_{ini}$  such that  $x^* = \Gamma(p^*)$ . Hence, we get the shooting parameters  $p^* = (\psi^*, \omega^*)$ . Moreover, by using the value function  $w$ , one can reconstruct an optimal trajectory and get the duration  $s_*$  of the first boost of the second engine, the duration  $\tau_B$  of the ballistic flight, and the control laws of the phases 1, 2 and 4.

## 5. NUMERICAL RESULTS

This section presents some numerical results obtained by using the ROC-HJ software (see Bokanowski et al. (2016b)) for solving HJB equations and computing optimal trajectories. The data used for the simulations was provided by CNES. The considered launcher is an Ariane-type two-stage spacecraft. Appart from the SSO target parameters, the numerical parameters used in the computations are similar to Bokanowski et al. (2015).

### 5.1 Numerical analysis of the problem

As in Bokanowski et al. (2015, 2016a) we use finite difference schemes for solving the HJB equation (23). In both articles the considered optimal control problem was related to the propellant consumption optimization for a fixed payload. In this new work the goal is to maximize the payload mass, and the problem is formulated for a hybrid dynamical system of switching type. Hence the optimal trajectory reconstruction procedure must be adapted to this new situation. Firstly, the value functions  $w_0$  and  $w$  are computed by solving the corresponding HJB equations.

Then, by using interpolation techniques, for each  $p = (\psi, \omega)$  in  $P_{ini}$  and corresponding point  $x = \Gamma(p) \in X_0$ , one computes an approximation of the function  $m^*(x)$ , as defined by (26). Figure 2 shows values of  $w(0, (\Gamma(p), m^*(\Gamma(p))))$  represented for different shooting parameters  $p = (\psi, \omega)$ .

One can therefore determine the optimal payload mass  $m_{opt}$  and the optimal shooting parameters by maximizing  $m^*(x)$  on  $X_0$ .

Finally, the optimal trajectory is computed using reconstruction algorithms (see for instance Assellaou et al. (2016)) starting from the optimal initial condition  $y^* = (x^*(p^*), m_{opt})$ . The reconstruction algorithm allows to compute not only the optimal control laws  $\mathbf{u}_1$  (from the value function  $w$ ) and  $\mathbf{u}_2$  (from the value function  $w_0$ )

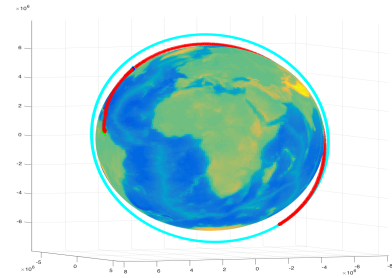


Fig. 3. Optimal trajectory (red) and the target SSO orbit (cyan), 3D view

Grid	Number of points	CPU (s)
Grid 1	$20 \times 30 \times 10 \times 10 \times 8 \times 3$	900
Grid 2	$30 \times 40 \times 15 \times 15 \times 12 \times 4$	3520
Grid 3	$40 \times 60 \times 20 \times 20 \times 16 \times 5$	18900

Table 1. Grid sizes and CPU times

Grid	$\psi$ (deg)	$\omega$ (deg s <sup>-1</sup> )	$s_*$ (s)	$\tau_B$ (s)	$m_{opt}$ (kg)
Grid 1	105.00	0.69	956.15	2605.82	15449.90
Grid 2	105.00	0.69	956.44	2625.82	15563.13
Grid 3	103.99	0.69	955.41	2605.82	15624.87

Table 2. Optimal initial parameters, phase durations and payload mass

but also the optimal duration of the ballistic flight  $\tau^*$  (considered here as a control of the state transfer switch).

Different computational grids for the HJB computations were tested to compare the results. Table 1 defines three grids and gives corresponding CPU times needed for the full HJB computations, the optimal payload and shooting parameters computations, and the reconstruction of the optimal trajectory. Computations were performed on processors Intel XEON E5-2695 v2, 2.4 GHz, using OPEN-MP version of the library ROC-HJ, with 30 threads.

Table 2 reports the main outputs from the computations: the initial parameters  $(\psi, \omega)$ , the optimal payload mass  $m_{opt}$ , the duration of the first boost  $s_*$  and the duration of the ballistic flight  $\tau_B$ . The optimal payload mass found with the finest grid is  $m_{opt} = 15624.87$  kg. We had no reference result for this problem. One can remark that this result is coherent with the informations provided in the user's manual of Ariane 5 Perez (2011).

One difficulty in this work was to optimize the payload mass and the ascending trajectory without any a priori information about the transfer orbit. These information can be obtained once the optimal trajectory is computed. In the table 3 are given the transfer orbital parameters obtained for the finest grid (Grid 3). The parameters are: the perigee argument  $\nu$ , the perigee altitude  $r_p$ , the apogee altitude  $r_a$ , and the inclination  $i$ . The figures 3 and 4 show the obtained optimal two-boost trajectory, starting on the Earth and arriving on the SSO with the optimal payload mass  $m_{opt} = 15624.87$  kg.

## 6. CONCLUSION

We have presented here some new results about the application of the HJB theoretical framework to the trajectory



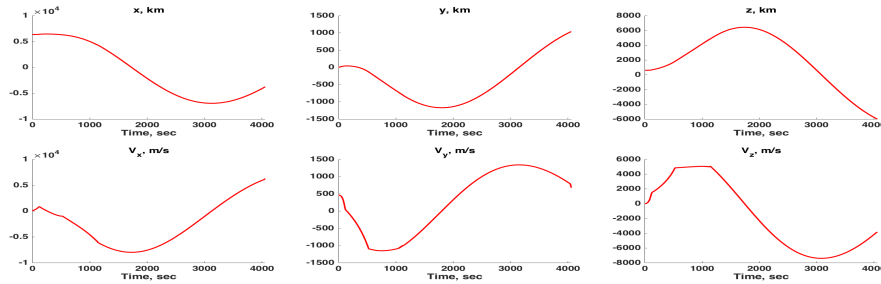


Fig. 4. Optimal trajectory (red), represented in the inertial cartesian reference frame

$\nu$ (deg)	$r_a$ (km)	$r_p$ (km)	$i$ (deg)
68.98	780.66	133.36	100.28

Table 3. Optimal transfer orbit parameters obtained with Grid 3

optimization problem for multi-stage space launchers. The most important challenge of this work was to design an optimal trajectory reconstruction procedure based on two different HJB equations to take into account the particular two-boost mission specifications.

#### Appendix A. ORBITAL PARAMETERS

Let  $(\vec{R}, \vec{V})$  be the state of the launcher in the inertial cartesian reference frame. Denoting  $r = \|\vec{R}\|$  and  $v = \|\vec{V}\|$ , the eccentricity  $e(x)$ , the major semi-axis  $a(x)$  and the inclination of the orbit  $i(x)$  are given by:

$$e(x) := \left\| \frac{\vec{V} \wedge (\vec{R} \wedge \vec{V})}{\mu} - \frac{\vec{R}}{r} \right\|$$

$$a(x) := \left| \frac{\mu}{2} \left( \frac{v^2}{2} - \frac{\mu}{r} \right)^{-1} \right| \quad \text{and} \quad i(x) := \arccos \left( \frac{(\vec{R} \wedge \vec{V})_3}{\|\vec{R} \wedge \vec{V}\|} \right)$$

These orbital parameters can be shown to be independent of the  $L$  coordinate (see Mooij (1994) for standard coordinate transformations between different reference frames).

#### REFERENCES

- Altarovici, A., Bokanowski, O., and Zidani, H. (2013). A general hamilton-jacobi framework for non-linear state-constrained control problems. *ESAIM: Control, Optimisation and Calculus of Variations*, 19, 337–357. doi: 10.1051/cocv/2012011.
- Assellaou, M., Bokanowski, O., Desilles, A., and Zidani, H. (2016). HJB approach for state constrained control problems with maximum cost. *Preprint*.
- Bardi, M. and Capuzzo-Dolcetta, I. (1997). *Optimal Control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Birkhäuser Boston.
- Bérend, N., Bonnans, J., Laurent-Varin, J., Haddou, M., and Talbot, C. (2005). An interior point approach to trajectory optimization. *J. of Guidance Control and Dynamics*, 30(5), 1228–1238.
- Betts, J. (2001). *Practical methods for optimal control using nonlinear programming*. Society for Industrial and Applied Mathematics, Philadelphia.
- Bokanowski, O., Bourgeois, E., Désilles, A., and Zidani, H. (2015). Global optimization approach for the climbing problem of multi-stage launchers. URL <https://hal-ensta.archives-ouvertes.fr/hal-01113819>. Preprint.
- Bokanowski, O., Bourgeois, E., Désilles, A., and Zidani, H. (2016a). HJB approach for a multi-boost launcher trajectory optimization problem. *Proceeding of IFAC 2016*.
- Bokanowski, O., Désilles, A., Zidani, H., and Zhao, J. (2016b). Roc-hj software. URL <http://uma.ensta-paristech.fr/soft/ROC-HJ/>.
- Bokanowski, O., Forcadel, N., and Zidani, H. (2010). Reachability and minimal times for state constrained nonlinear problems without any controllability assumption. *SIAM Journal on Control and Optimization*, 48(7), 4292–4316.
- Bonnans, F., Martinon, P., and Trélat, E. (2008). Singular arcs in the generalized Goddard’s problem. *J. Optimization Theory Applications*, 139 (2), 439–461.
- Bourgeois, E., Bokanowski, O., Désilles, A., and Zidani, H. (2015). Optimization of the launcher ascent trajectory leading to the global optimum without any initialization: the breakthrough of the hjb approach. In *6th European Conference for Aeronautics and Space Sciences (EUCASS)*.
- Mooij, E. (1994). The motion of a vehicle in a planetary atmosphere. Technical report, TU Delft.
- Oberle, H. (1990). Numerical computation of singular control functions in trajectory optimization problems. *Journal of Guidance, Control and Dynamics*, 13, 153–159.
- Perez, E. (2011). *Ariane 5. User’s manual. Issue 5. Revision 1*. Arianespace.
- Ponssard, C., Graichen, K., Petit, N., and Laurent-Varin, J. (2009). Ascent optimization for a heavy space launcher. In *European Control Conference (ECC), Budapest, Hungary*.
- Zondervan, K., Bauer, T., Betts, J., and Huffman, W. (1984). Solving the optimal control problem using a nonlinear programming technique. part 3: Optimal shuttle reentry trajectories. *Proceedings of the AIAA/AAS Astrodynamics conference, Seattle*.