Error estimates for numerical approximation of Hamilton-Jacobi equations related to hybrid control systems

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Abstract

Hybrid control systems are dynamical systems that can be controlled by a combination of both continuous and discrete actions. In this paper we study the approximation of optimal control problems associated to this kind of systems, and in particular of the Quasi-Variational Inequality which characterizes the value function. Our main result features the error estimates between the value function of the problem and its approximation. We also focus on the hypotheses describing the mathematical model and the properties defining the class of numerical scheme for which the result holds true.

Keywords: Hybrid control, Dynamic Programming, Semi-Lagrangian schemes, Error estimates

AMS subject classification: 34A38, 49L20, 49M25, 49N25, 65K15

1 Introduction

Hybrid control systems are described by a combination of continuous and discrete or logical variables and have been the subject of much attention over the last decade. A classical example of a hybrid control problem is the model of a vehicle equipped with two engines: an electric engine (EE) and an internal combustion engine (ICE). The former is powered by a battery that is recharged by the latter, which instead consumes regular fuel. In an optimal control problem for such a system, the goal might be to minimize a combination of fuel consumption and speed by acting on both the acceleration strategy and the discontinuous switching between EE and ICE.

The mathematical formulations of optimal control problems for hybrid systems we will adopt in this paper is the one given in [10, 7, 2]. We focus on the infinite horizon hybrid control problem, whose value function and numerical approximation have already been studied in [12], with the aim of estimating the numerical error in the approximation of the value function. For the theoretical analysis of the numerical scheme, we put ourselves in the framework introduced by Barles and Souganidis [4], which allows to treat various approximation schemes like Finite Difference methods [17, 18], Semi-Lagrangian schemes [11, 8] and Markov chain approximations [17]. The main tool used to estimate the numerical error in the approximation of the value function will be a technique based on the *shaking coefficients* method introduced by Krylov in [15, 16]; we will also make use of

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the *cascade* technique, which consists in approaching the main problem by a sequence of obstacle problems, through an adaptation of the arguments in [13].

The outline of the paper is the following. In Section 2 we will set the basic assumptions on the control problem and review the characterization of the value function in terms of a suitable Dynamic Programming equation. In Section 3 we will study the numerical approximation via monotone schemes and discuss convergence and solvability of the numerical scheme. Section 4 describes the cascade technique, used to obtain the estimates for intermediate problems that will be applied in order to prove the main result. In Sections 5 and 6 we prove some regularity properties for the value functions of the problems generated by the cascade and give the error estimates between the exact solutions and their approximations. Lastly, we collect in Appendix A the proofs of some auxiliary results stated throughout the paper.

2 Preliminaries

We start by introducing some notations. We denote by $|\cdot|$ the standard Euclidean norm in any $\mathbb{R}^d$ type space (for any $d \geq 1$). If B is a $d \times d$ matrix, then $|B|^2 = tr(BB^\intercal)$, where $B^\intercal$ is the transpose of B and $|B|$ is the Frobenius norm. For a discrete set S, $|S|$ will denote its cardinality.

Let ϕ be a bounded function from $\mathbb{R}^d$ into either $\mathbb{R}$, $\mathbb{R}^d$, or the space of $d \times m$ matrices ($m \geq 1$). We define

$$|\phi|_0 := \sup_{x \in \mathbb{R}^d} \big|\phi(x)\big|.$$

If ϕ is also Lipschitz continuous, we set

$$|\phi|_1 := \sup_{x,y \in \mathbb{R}^d, x \neq y} \frac{\big|\phi(x) - \phi(y)\big|}{|x-y|}$$

Moreover, for any closed set $\mathcal{S} \subset \mathbb{R}^d$, the space $C_{\mathrm{b}}(\mathcal{S})$ [respectively, $C_{\mathrm{b,l}}(\mathcal{S})$] will denote the space of continuous and bounded functions [resp., bounded and Lipschitz continuous functions] from $\mathcal{S}$ to $\mathbb{R}$.

Given $\phi \in \big[C_{\mathrm{b,l}}(\mathbb{R}^d)\big]^m$, we denote by L_ϕ and M_ϕ some *upper bounds* of respectively the Lipschitz constant and the supremum of ϕ:

$$L_\phi \geq \max_{i \in \{1,\dots,m\}} |\phi_i|_1 \qquad M_\phi \geq |\phi|_0.$$

We denote by $\leq$ the componentwise ordering in $\mathbb{R}^d$, and by $\preceq$ the ordering in the sense of positive semi-definite matrices. For any $a, b \in \mathbb{R}$, we define $a \wedge b$ as

$$a \wedge b := \min(a, b).$$

For any given closed subset $\mathcal{S}$ of $\mathbb{R}^d$, the notations $\partial\mathcal{S}$, $\mathrm{dist}(\cdot, \mathcal{S})$ stand respectively for the boundary of $\mathcal{S}$ and the Euclidean distance defined by

$$\mathrm{dist}(x, \mathcal{S}) := \inf_{y \in \mathcal{S}} |x - y|.$$

Among the various mathematical formulations of optimal control problems for hybrid systems, we will adopt here the one given in [10, 7, 2]. Let therefore $\mathbb{I}$ be a finite set, and consider the controlled system (X, Q) satisfying:

$$\begin{cases} \dot{X}(t) = f\big(X(t), Q(t), u(t)\big) \\ X(0) = x \\ Q(0^+) = q \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^d$, and $q \in \mathbb{I}$. Here, X and Q represent respectively the continuous and the discrete component of the state. Note that throughout the paper we will term *switch* a transition in the state which involves only a change in the $Q(t)$ component, whereas *jump* will denote a transition which might also involve a discontinuous change in $X(t)$.

The function $f : \mathbb{R}^d \times \mathbb{I} \times \mathcal{U} \to \mathbb{R}^d$ is the continuous dynamics and the continuous control set is:

$$\mathcal{U} = \big\{u : (0, +\infty) \to U \quad | \quad u \text{ measurable, } U \text{ compact metric space}\big\}.$$

The trajectory may undergo discrete transitions when it evolves inside the set $\mathbb{R}^d \times \mathbb{I}$. More precisely, the controller can choose either to jump or not.

If the controller chooses to jump, then the continuous trajectory is moved to a new point in $\mathcal{D} \subset \mathbb{R}^d \times \mathbb{I}$. By ξ_i we denote a transition time. The state $\big(X(\xi_i^-), Q(\xi_i^-)\big)$ is moved by the controlled jump the to the destination $\big(X(\xi_i^+), Q(\xi_i^+)\big) \in \mathcal{D}$. The trajectory starting from $x \in \mathbb{R}^d$ with discrete state $q \in \mathbb{I}$ is therefore composed of a continuous evolution given by (2.1) between two discrete jumps at the transition times. For example, for $\xi_k < t < \xi_{k+1}$, the evolution of the hybrid system would be given by:

$$\begin{cases} \big(X(\xi_k^+), Q(\xi_k^+)\big) \in \mathcal{D} & \text{(destination of the jump at } \xi_k) \\ \dot{X}(t) = f\big(X(t), Q(\xi_k^+), u(t)\big) & \xi_k < t < \xi_{k+1} \end{cases}$$

Associated to this hybrid system, we consider an infinite horizon control problem where the cost is composed of a running cost and transition costs corresponding to the controlled and uncontrolled jumps. A similar control problem has been considered in [12], where the authors have studied the value function and its numerical approximation. A procedure to compute a piecewise constant feedback control is also analyzed in [12].

2.1 Basic assumptions

In the product space $\mathbb{R}^d \times \mathbb{I}$, we consider the set $\mathcal{D}$ in the form

$$\mathcal{D} = \big\{(x, q) \in \mathbb{R}^d \times \mathbb{I} : x \in D_q\big\} \tag{2.2}$$

in which $\mathcal{D}_i$ represents the subset of $\mathcal{D}$ in which $q = i$.

We make the following standing assumptions on the set $\mathcal{D}$ and on the functions f and g:

(A1) For each $q \in \mathbb{I}$, $\mathcal{D}_q$ is a closed subset of $\mathbb{R}^d$, and $\mathcal{D}_q$ is bounded.

This assumption is essential to the well-posedness of the HJB equation resulting from the characterization of the value function.

(A2) The function f is Lipschitz continuous with Lipschitz constant L_f in the state variable x and uniformly continuous in the control variable u. Moreover, for all $(x, q) \in \mathbb{R}^d \times \mathbb{I}$ and $u \in U$,

$$\big|f(x, q, u)\big| \leq M_f$$

This hypothesis could be replaced by the less strict requirement of f being only locally Lipschitz, in order to extend the results to a more general case. However, the boundedness of f is a standard simplification in the framework of hybrid optimal control problems (see [1] and [2]).

In what follows, a control policy for the hybrid system consists of two parts: continuous input u and discrete inputs. A continuous control is a measurable function $u \in \mathcal{U}$ acting on the trajectory through the continuous dynamics (2.1). The discrete inputs take place at the transition times

$$0 \leq \xi_0 \leq \xi_1 \leq \cdots \leq \xi_k \leq \xi_{k+1} \leq \ldots$$

in which at time ξ_k the trajectory moves to a new position $(x'_k, q'_k) \in \mathcal{D}$. The discrete input is therefore in the from $\{(\xi_k, x'_k, q'_k)\}_{k\geq 0}$. To shorten the notation, we will denote by $\theta := \Big(u(\cdot), \big\{(\xi_k, x'_k, q'_k)\big\}\Big)$ a hybrid control strategy, and by Θ the set of all admissible strategies.

Now, for every control strategy $\theta \in \Theta$, we associate the cost defined by:

$$J(x, q; \theta) := \int_0^{+\infty} \ell\big(X(t), Q(t), u(t)\big)e^{-\lambda t}dt + \sum_{k=0}^{\infty} c\big(X(\xi_k^-), Q(\xi_k^-), X(\xi_k^+), Q(\xi_k^+)\big)e^{-\lambda \xi_k} \tag{2.3}$$

where $\lambda > 0$ is the discount factor, $\ell : \mathbb{R}^d \times \mathbb{I} \times U \to \mathbb{R}_+$ is the running cost and $c : \mathbb{R}^d \times \mathbb{I} \times \mathcal{D} \to \mathbb{R}_+$ is the controlled transition cost. The value function V is then defined as:

$$V(x, q) := \inf_{\theta \in \Theta} J(x, q; \theta). \tag{2.4}$$

We assume the following conditions on the cost functional:

(A3) $\ell : \mathbb{R}^d \times \mathbb{I} \times U \to \mathbb{R}$ is a bounded and nonnegative function, Lipschitz continuous with respect to the x variable, and uniformly continuous w.r.t. the u variable.

(A4) $c : \mathbb{R}^d \times \mathbb{I} \times \mathcal{D} \to \mathbb{R}$ is bounded with a strictly positive infimum $K_0 > 0$ and uniformly Lipschitz continuous in the variable x'.

(A5) The discount factor λ satisfies $\lambda > \max(1, L_f)$.

2.2 Characterization of the value function

We briefly review the main theoretical facts about the value function V defined in (2.4).

It is quite straightforward to derive the Dynamic Programming Principle for the control problem (2.1)–(2.3). For any $(x, q) \in \mathbb{R}^d \times \mathbb{I}$ there exists $s_0 > 0$ such that, for every $0 < s < s_0$, we have

$$V(x, q) \leq \inf_{(x',q') \in \mathcal{D}} \big\{V(x', q') + c(x, q, x', q')\big\}. \tag{2.5}$$

If it happens that $V(x, q) < \inf_{(x',q')\in\mathcal{D}} \big\{V(x', q') + c(x, q, x', q')\big\}$, then there exists $s_0 > 0$ such that for every $0 < s < s_0$, we have:

$$V(x, q) = \inf_{u \in \mathcal{U}} \left\{ \int_0^s \ell\big(X(t), q, u(t)\big)e^{-\lambda t}\,dt + e^{-\lambda s}V\big(X(s), q\big) \right\} \tag{2.6}$$

Moreover, it is known that the value function V is uniformly continuous [10, Theorem 3.5]. More precisely, we have:

Lemma 2.1. *Under assumptions (A1)–(A4), the function V is bounded and Hölder continuous.*

From the dynamic programming principle, it can be checked that the value function satisfies, in an appropriate sense, a quasi-variational inequality. To give a precise statement of this result, we first introduce the Hamiltonian $H : \mathbb{R}^d \times \mathbb{I} \times \mathbb{R}^d \to \mathbb{R}$ defined, for $x, p \in \mathbb{R}^d$ and $q \in \mathbb{I}$, by:

$$H(x, q, p) := \sup_{u \in U} \big\{ -\ell(x, q, u) - f(x, q, u) \cdot p \big\} \tag{2.7}$$

We also define the transition operator $\mathcal{N}$, mapping $C^0(\mathbb{R}^d \times \mathbb{I})$ into itself, by:

$$\mathcal{N}\phi(x,q) := \inf_{(x',q')\in\mathcal{D}} \left\{\phi(x',q') + c(x,q,x',q')\right\} \quad (x,q) \in \mathbb{R}^d \times \mathbb{I} \tag{2.8}$$

The following properties hold for $\mathcal{N}$.

Proposition 2.2. *Let $\phi, \psi : \mathbb{R}^d \times \mathbb{I} \to \mathbb{R}$ and $\mathcal{N}$ be defined by* (2.8)*. Then:*

1. *If $\phi \leq \psi$, then $\mathcal{N}\phi \leq \mathcal{N}\psi$*
2. $\mathcal{N}(t\phi + (1-t)\psi) \geq t\mathcal{N}\phi + (1-t)\mathcal{N}\psi \quad \forall t \in [0,1]$
3. $\mathcal{N}(\phi + c) = \mathcal{N}\phi + c \quad \forall c \in \mathbb{R}$

Remark 2.3. *These properties are similar to the ones from [13] and follow from the definition of $\mathcal{N}$.*

Now we go back to the characterization of the value function V. It turns out that V solves the Hamilton–Jacobi–Bellman (HJB) equation

$$\max\left\{\lambda V(x,q) + H\big(x,q,D_x V(x,q)\big), V(x,q) - \mathcal{N}V(x,q)\right\} = 0 \quad (x,q) \in \mathbb{R}^d \times \mathbb{I} \tag{2.9}$$

provided the solution is understood in the *viscosity* sense used in [5].

Definition 2.4 (Viscosity solution)**.** *Assume (A1)–(A4). Let $w : \mathbb{R}^d \times \mathbb{I} \to \mathbb{R}$ be a bounded and uniformly continuous function. We say that w is a viscosity sub-[respectively, super-]solution of the HJB equation* (2.9) *if, for any bounded function $\phi : \mathbb{R}^d \to \mathbb{R}$ with continuous and bounded first derivative, the following property holds.*

For any $q \in \mathbb{I}$, at each local maximum [resp., minimum] point (x', q) of $w(x,q) - \phi(x)$ we have

$$\max\Big\{\lambda V(x',q) + H\big(x',q,D_x\phi(x')\big), V(x',q) - \mathcal{N}V(x',q)\Big\} \leq 0 \quad [\textit{resp.} \geq 0] \quad (x,q) \in \mathbb{R}^d \times \mathbb{I}$$

A viscosity solution is a function which is simultaneously sub- and super-solution.

The previous definition allows us to characterize V.

Proposition 2.5. *Assume (A1)–(A4). Then, the function V is a bounded and Hölder continuous viscosity solution of* (2.9)*.*

The proof is given in [10, Theorem 3.5]. The same arguments of the proof of Theorem 5.1 in [10] can then be used to obtain a strong comparison principle (and hence, uniqueness of the solution) as follows:

Theorem 2.6. *Assume (A1)–(A5). Let w [respectively, v] be a bounded usc [resp., lsc] function on $\mathbb{R}^d$. Assume that w is a sub-solution [resp., v is a super-solution] of* (2.9) *in the following sense: for any $q \in \mathbb{I}$*

$$\max\Big\{\lambda V(x,q) + H\big(x,q,D_x V(x,q)\big), V(x,q) - \mathcal{N}V(x,q)\Big\} \leq 0 \quad [\textit{resp.} \geq 0] \quad (x,q) \in \mathbb{R}^d \times \mathbb{I}$$

Then, $w \leq v$.

Recall that the viscosity framework turns out to be a convenient tool for the study of both the theoretical properties of the value function and the convergence of numerical schemes.

3 The numerical scheme

In this section, we review the basic theory of convergence for monotone numerical approximations. The approximation of the value function is obtained by a suitable adaptation of monotone schemes to the hybrid case. The final goal of proving error estimates for this approximation will borrow some ideas and techniques introduced in [15, 16], as well as a sensitivity analysis of the value function with respect to perturbations of the trajectories.

Consider monotone approximation schemes of (2.9), of the following form:

$$\max\Big\{S\big(h,x,q,V_h(x,q),V_h\big),V_h(x,q)-\mathcal{N}V_h(x,q)\Big\}=0\quad (x,q)\in\mathbb{R}^d\times\mathbb{I}. \tag{3.1}$$

Here, $S:\mathbb{R}^d_+\times\mathbb{R}^d\times\mathbb{I}\times\mathbb{R}\times C_{\mathrm{b}}(\mathbb{R}^d\times\mathbb{I})\to\mathbb{R}$ is a consistent, monotonic operator which is considered to be an approximation of the HJB equation (2.9) (see assumptions (S1)–(S4) for the precise properties). We will denote by $h\in\mathbb{R}^d_+$ the mesh size, and by $V_h\in C_{\mathrm{b}}(\mathbb{R}^d\times\mathbb{I})$ the solution of (3.1).

The abstract notations of the scheme was introduced by Barles and Souganidis [4] to display the monotonicity of the scheme: $S(h,x,q,r,v)$ is non decreasing in r and non increasing in v. Typical approximation schemes that can be put in this framework are finite differences methods [17, 18], Semi-Lagrangian schemes [11, 8], and Markov chain approximations [17]. In all the sequel, we make the following assumptions on the discrete scheme (3.1):

(S1) Monotonicity: for all $h\in\mathbb{R}^d_+$, $m\geq 0$, $x\in\mathbb{R}^d$, $q\in\mathbb{I}$, $r\in\mathbb{R}$, and ϕ,ψ in $C_{\mathrm{b}}(\mathbb{R}^d)$ such that $\phi\leq\psi$ in $\mathbb{R}^d$

$$S(h,x,q,r,\phi+m)\geq m+S(h,x,q,r,\psi)$$

(S2) Regularity: for all $h\in\mathbb{R}^d_+$ and $\phi\in C_{\mathrm{b}}(\mathbb{R}^d)$, $x\mapsto S(h,x,q,r,\phi)$ is bounded and continuous. For any $R>0$, $r\mapsto S(h,x,q,r,\phi)$ is uniformly continuous on the ball $B(0,R)$ centered at 0 and with radius R, uniformly with respect to $x\in\mathbb{R}^d$.

(S3) Consistency: There exist $p,k_i>0$, $i\in J\subseteq\{1,\ldots,p\}$ and a constant $K_c>0$ such that, for all $h\in\mathbb{R}^d_+$ and x in $\mathbb{R}^d$, and for every smooth $\phi\in C^p(\mathbb{R}^d)$ such that $|D^i\phi|_0$ is bounded, for every $i\in J$ and $q\in\mathbb{I}$, the following holds:

$$\Big|\lambda\phi(x)+H\big(x,q,D_x\phi(x)\big)-S\big(h,x,q,\phi(x),\phi\big)\Big|\leq K_{\mathrm{c}}\mathcal{E}(h,\phi),$$

where $\mathcal{E}(h,\phi):=\sum_{i\in J}|D^i\phi|_0|h|^{k_i}$. Here, $D^i\phi$ denotes the i-th derivative of the function ϕ.

(S4) Let $\eta\geq 0$ be a constant. If v is solution of

$$\max\Big\{S\big(h,x,q,v(x,q),v\big),v(x,q)-\mathcal{N}v(x,q)\Big\}=0,$$

then $v+\eta$ is solution of

$$\max\Big\{S\big(h,x,q,v(x,q),v\big)+\eta\lambda;v(x,q)-\mathcal{N}v(x,q)\Big\}=0.$$

Moreover, if S can be written in the form

$$S(h,x,q,r,\phi)=\max_{u\in U}S^u\big(h,x,q,r,\phi\big),$$

then, for $\mu\in(0,1)$, μv is a sub-solution of

$$\max\Big\{\max_{u\in U}S^u\big(h,x,q,\mu v(x,q)+(\mu-1)\ell(x,q,u),\mu v\big),\mu v(x,q)-\mu\mathcal{N}v(x,q)\Big\}\leq 0 \tag{3.2}$$

The adaptation of classical monotone schemes to the Bellman equation (2.9) has been studied in [12]. In particular, this work proves convergence for iterative solvers based on value iteration, as well as the properties of consistency, monotonicity and L^∞ stability, which imply convergence of the approximate value function via the Barles-Souganidis theorem.

4 Cascade Problems

The main goal of this paper is to derive error estimate between the value function V and its approximation V_h. The main difficulties in this study come from the presence of controlled jumps, which introduce coupling terms (represented by the highly nonlinear operator $\mathcal{N}$) in the HJB equation. To deal with these difficulties, we will use an idea of cascade problems, described in the following subsections.

4.1 Cascade for the HJB equation

We approach equation (2.9) by a sequence of obstacle problems, and use the same methods as in [13, Proof of Theorem 4.2], to prove that the related sequence of solutions converges to the solution of (2.9). Consider the problem:

$$\lambda V_0(x,q) + H\big(x,q,D_xV_0(x,q)\big) = 0 \quad (x,q) \in \mathbb{R}^d \times \mathbb{I}. \tag{4.1}$$

Under assumptions (A1)–(A2), this equation has a unique viscosity solution V_0 in $C_{\mathrm{b,l}}(\mathbb{R}^d \times \mathbb{I})$. Since $V \equiv 0$ is a viscosity sub-solution of (4.1), the comparison principle (see [13, Theorem 3.3]) implies $0 \leq V_0$. Now, for a given V_{n-1} in $C_{\mathrm{b,l}}(\mathbb{R}^d \times \mathbb{I})$ and $n \geq 1$, consider the problem:

$$\max\Big\{\lambda V_n(x,q) + H\big(x,q,D_xV_n(x,q)\big), V_n(x,q) - \mathcal{N}V_{n-1}(x,q)\Big\} = 0 \quad (x,q) \in \mathbb{R}^d \times \mathbb{I}. \tag{4.2}$$

Note that (4.1)–(4.2) is *not* the form in which the actual numerical solution is computed, but rather an auxiliary family of problems, in which the value function V_n corresponds to optimal solutions which satisfy the constraint of performing *at most n jumps in the state.*

Since $\mathcal{N}V_{n-1}$ is uniformly continuous, under assumptions (A1)–(A2), there exists a unique viscosity solution V_n of (4.2) in $C_{\mathrm{b,l}}(\mathbb{R}^d \times \mathbb{I})$. It is easy to check that V_1 is a viscosity sub-solution of (4.1). By the comparison principle, $V_1 \leq V_0$. Moreover, $V \equiv 0$ is a sub-solution of (4.2) for $n = 1$, and then $0 \leq V_1 \leq V_0$ in $\mathbb{R}^d$. By point (1) of Proposition 2.2 $\mathcal{N}V_1 \leq \mathcal{N}V_0$, so tha we can say that V_2 is a viscosity sub-solution of (4.2) for $n = 1$, and also $V_2 \leq V_1$ in $\mathbb{R}^d \times \mathbb{I}$.

By induction over n, we obtain:

$$0 \leq \cdots \leq V_n \leq \cdots \leq V_2 \leq V_1 \leq V_0. \tag{4.3}$$

We can see that, if $|V_0|_0 \leq K_0$ (where K_0 is defined in assumption (A4)), then $V = V_0$ is a viscosity solution of (4.1). Intuitively, this corresponds to the situation in which optimal solutions of the control problem do not perform jumps in the state. In this case, we refer to §6.2 for the specific error estimate.

Suppose now that $|V_0|_0 > K_0$, and let $\mu \in (0,1)$ such that $\mu|V_0|_0 < K_0$.

Theorem 4.1. *We have that, for all n,*

$$V_n - V_{n+1} \leq (1-\mu)^n |V_0|_0. \tag{4.4}$$

Moreover, V_n converges towards V, when n tends to $+\infty$ and

$$0 \leq V_n - V \leq \frac{(1-\mu)^n}{\mu} |V_0|_0. \tag{4.5}$$

Proof. The same arguments used in [13, Proof of Theorem 4.2] can be used here. For reader's convenience, we repeat here the main steps. Let $n \in \mathbb{N}$, and $\theta_n \in (0,1]$ be such that, in $\mathbb{R}^d \times \mathbb{I}$

$$V_n - V_{n+1} \leq \theta_n V_n. \tag{4.6}$$

By (4.3), this holds at least for $\theta_n = 1$. Rewriting (4.6) as $(1-\theta_n)V_n \leq V_{n+1}$, and using Proposition 2.2 and assumption (A4), we get

$$(1-\theta_n)\mathcal{N}V_n + \theta_n K_0 \leq (1-\theta_n)\mathcal{N}V_n + \theta_n \mathcal{N}0 \leq \mathcal{N}\big((1-\theta_n)V_n\big) \leq \mathcal{N}V_{n+1}. \tag{4.7}$$

We now prove that

$$(1-\theta_n+\mu\theta_n)V_{n+1} \leq V_{n+2} \tag{4.8}$$

where V_{n+2} solves (4.2) at the step $n+2$. Since V_{n+1} solves (4.2) at the step $n+1$, and $\ell(x,q,u) \geq 0$ for all x, q and u, we have that $(1-\theta_n+\mu\theta_n)V_{n+1}$ is a viscosity sub-solution of

$$\max\Big\{\lambda V(x,q) + H\big(x,q,D_x V(x,q)\big), V(x,q) - \mathcal{N}V(x,q)\Big\} \leq 0 \qquad (x,q) \in \mathbb{R}^d \times \mathbb{I}.$$

Moreover, by the construction of the sequence (4.3), and by (4.7), we have

$$(1-\theta_n+\mu\theta_n)V_{n+1} \leq (1-\theta_n)V_{n+1} + \mu\theta_n|V_0|_0 \tag{4.9}$$

$$\mathcal{N}V_{n+1} \geq (1-\theta_n)\mathcal{N}V_n + \theta_n K_0. \tag{4.10}$$

Taking the difference between (4.9) and (4.10), and knowing that V_{n+1} is the viscosity solution of (4.2), we have

$$\begin{aligned}
&(1-\theta_n+\mu\theta_n)V_{n+1} - \mathcal{N}V_{n+1} \leq \\
&\quad\leq (1-\theta_n)V_{n+1} + \mu\theta_n|V_0|_0 - (1-\theta_n)\mathcal{N}V_n - \theta_n K_0 \leq \\
&\quad\leq (1-\theta_n)V_{n+1} + \theta_n K_0 - (1-\theta_n)\mathcal{N}V_n - \theta_n K_0 \leq 0
\end{aligned}$$

which implies

$$(1-\theta_n+\mu\theta_n)V_{n+1} - \mathcal{N}V_{n+1} \leq 0.$$

Then, we can infer that $(1-\theta_n+\mu\theta_n)V_{n+1}$ is a viscosity sub-solution of (4.2) at the step $n+2$. The comparison principle implies (4.8), or equivalently

$$V_{n+1} - V_{n+2} \leq \theta_n(1-\mu)V_{n+1}. \tag{4.11}$$

By the inequalities $V_0 - V_1 \leq V_0$ in $\mathbb{R}^d \times \mathbb{I}$, we obtain $V_1 - V_2 \leq (1-\mu)V_1$ in $\mathbb{R}^d \times \mathbb{I}$. Then, comparing (4.11) and (4.6), it follows that θ_n can be defined as $\theta_n = (1-\mu)^n$, so that

$$V_{n+1} - V_{n+2} \leq (1-\mu)^{n+1}V_{n+1} \leq (1-\mu)^{n+1}|V_0|_0. \tag{4.12}$$

By (4.3) and (4.4), we can find a function $V \in C(\mathbb{R}^d \times \mathbb{I})$, such that $|V_n - V|_0 \to 0$, when $n \to +\infty$. Proposition 2.2 and the stability of solutions imply that V is a viscosity solution of (2.9). Then we can say that V_n converges to V, the unique viscosity solution of (2.9), when $n \to +\infty$. Moreover, by (4.4) and since $(1-\mu) < 1$, the following upper bound holds in $\mathbb{R}^d \times \mathbb{I}$ for all $n \geq 0$

$$0 \leq V_n - V \leq \sum_{i=n}^{+\infty}(1-\mu)^i|V_0|_0 = \frac{(1-\mu)^n}{1-(1-\mu)}|V_0|_0 = \frac{(1-\mu)^n}{\mu}|V_0|_0. \tag{4.13}$$

$\square$

4.2 Cascade for the numerical scheme

As we have done for the equation (2.9), we will approach (3.1) by a sequence of equations approximating (4.2).

Let $V_{h0} \in C_{\mathrm{b}}(\mathbb{R}^d \times \mathbb{I})$ be a solution of

$$S\big(h, x, q, V_{h0}(x,q), V_{h0}\big) = 0 \quad (x,q) \in \mathbb{R}^d \times \mathbb{I} \tag{4.14}$$

Define $V_{h1} \in C_{\mathrm{b}}(\mathbb{R}^d \times \mathbb{I})$ a solution of the problem:

$$\max\Big\{S\big(h, x, q, v(x,q), v\big), v(x,q) - \mathcal{N}V_{h0}(x,q)\Big\} = 0 \quad (x,q) \in \mathbb{R}^d \times \mathbb{I}. \tag{4.15}$$

For $n = 2, 3, \cdots$, we consider the family of solutions V_{hn} of

$$\max\Big\{S\big(h, x, q, v(x,q), v\big), v(x,q) - \mathcal{N}V_{h(n-1)}(x,q)\Big\} = 0 \quad (x,q) \in \mathbb{R}^d \times \mathbb{I}. \tag{4.16}$$

In order to prove the results in this section, we make the following assumptions.

($\tilde{\mathbf{S}}$1) For every sufficiently small $h > 0$ and $n \geq 1$ the solutions V_{hn} of (4.14) and (4.16) exist.

($\tilde{\mathbf{S}}$2) The value functions V_{hn} are Lipschitz continuous for every $n \geq 0$ and their Lipschitz constants satisfy

$$0 \leq \cdots \leq L_{V_{hn}} \leq \cdots \leq L_{V_{h2}} \leq L_{V_{h1}} \leq L_{V_{h0}} \tag{4.17}$$

The function V_{h1} is a sub-solution of (4.14), and then $V_{h1} \leq V_{h0}$ in $\mathbb{R}^d \times \mathbb{I}$. Using proposition 2.2 and assumption (S4), one can verify that $V_h \equiv 0$ is a sub-solution of (4.15) in $\mathbb{R}^d \times \mathbb{I}$, which gives that $0 \leq V_{h1} \leq V_{h0}$ in $\mathbb{R}^d \times \mathbb{I}$. Proposition 2.2 implies that $0 \leq \mathcal{N}V_{h1} \leq \mathcal{N}V_{h0}$, therefore V_{h2} is a sub-solution of (4.15), and hence $V_{h2} \leq V_{h1}$ in $\mathbb{R}^d \times \mathbb{I}$. By induction on n,

$$0 \leq \cdots \leq V_{hn} \leq \cdots \leq V_{h2} \leq V_{h1} \leq V_{h0} \tag{4.18}$$

As in Subsection 4.1, we suppose that $|V_0|_0 > K_0$. Then, since $V_{h0} \to V_0$ uniformly (Barles-Souganidis Theorem), we have also $|V_{h0}|_0 > K_0$ for h small enough and we can choose $\mu \in (0,1)$ such that $\mu|V_0|_0 < K_0$, and $\mu|V_{h0}|_0 < K_0$.

Theorem 4.2. *Assume ($\tilde{S}$1). Then for all n and for h small enough, in $\mathbb{R}^d \times \mathbb{I}$ we have*

$$V_{hn} - V_{h(n+1)} \leq (1-\mu)^n |V_{h0}|_0 \tag{4.19}$$

Proof. We use the same methods as in Theorem 4.1, taking into account the monotonicity of S. □

Proposition 4.3. *Under assumptions (S1), (S4) and ($\tilde{S}$1), we have $|V_{hn} - V_h|_0 \to 0$ for $n \to +\infty$. Moreover, for $(x,q) \in \mathbb{R}^d \times \mathbb{I}$ and $n \geq 1$,*

$$V_{hn} - V_h \leq \sum_{i=n}^{+\infty} (1-\mu)^i |V_{h0}|_0 = \frac{(1-\mu)^n}{\mu} |V_{h0}|_0. \tag{4.20}$$

Proof. We follow similar steps as in the proof of Theorem 4.1.

Let $n \in \mathbb{N}$, and $\theta_n \in (0,1]$ be such that, in $\mathbb{R}^d \times \mathbb{I}$

$$V_{hn} - V_{h(n+1)} \leq \theta_n V_{hn}. \tag{4.21}$$

By (4.18), this holds at least for $\theta_n = 1$. Rewriting (4.21) as $(1-\theta_n)V_{hn} \leq V_{h(n+1)}$, and using Proposition 2.2, get

$$(1-\theta_n)\mathcal{N}V_{hn} + \theta_n K_0 \leq (1-\theta_n)\mathcal{N}V_{hn} + \theta_n \mathcal{N}0 \leq \mathcal{N}\big((1-\theta_n)V_{hn}\big) \leq \mathcal{N}V_{h(n+1)}. \tag{4.22}$$

We prove now that

$$(1-\theta_n+\mu\theta_n)V_{h(n+1)} \leq V_{h(n+2)} \tag{4.23}$$

where $V_{h(n+2)}$ is the solution of (4.16) at the step $n+2$.

Since $V_{h(n+1)}$ is the solution of (4.16) at the step $n+1$, assumption (S4) implies that $(1-\theta_n+\mu\theta_n)V_{h(n+1)}$ is a sub-solution of

$$\max\Big\{\max_{u\in U} S^u\big(h,x,q,\sigma_n V_{h(n+1)}(x,q) + (\sigma_n-1)\ell(x,q,u), \sigma_n V_{h(n+1)}\big), \sigma_n v(x,q) - \sigma_n \mathcal{N}v(x,q)\Big\} \leq 0.$$

with $\sigma_n := (1-\theta_n+\mu\theta_n)$. By (S1), we also have that

$$\begin{aligned}&\max_{u\in U} S^u\big(h,x,q,\sigma_n V_{h(n+1)}(x,q) + (\sigma_n-1)\ell(x,q,u), \sigma_n V_{h(n+1)}\big) \leq \\ &\quad \leq S\big(h,x,q,\sigma_n V_{h(n+1)}(x,q), \sigma_n V_{h(n+1)}\big) \leq 0.\end{aligned}$$

Moreover, by the construction of the sequence (4.18), and by (4.22), we obtain

$$(1-\theta_n+\mu\theta_n)V_{h(n+1)} \leq (1-\theta_n)V_{h(n+1)} + \mu\theta_n|V_{h0}|_0 \tag{4.24}$$

$$\mathcal{N}V_{h(n+1)} \geq (1-\theta_n)\mathcal{N}V_{hn} + \theta_n K_0. \tag{4.25}$$

Taking the difference between (4.24) and (4.25), and knowing that V_{n+1} is the solution of (4.16), we have

$$\begin{aligned}&(1-\theta_n+\mu\theta_n)V_{h(n+1)} - \mathcal{N}V_{h(n+1)} \leq \\ &\quad \leq (1-\theta_n)V_{h(n+1)} + \mu\theta_n|V_{h0}|_0 - (1-\theta_n)\mathcal{N}V_{hn} - \theta_n K_0 \leq \\ &\quad \leq (1-\theta_n)V_{h(n+1)} + \theta_n K_0 - (1-\theta_n)\mathcal{N}V_{hn} - \theta_n K_0 \leq 0\end{aligned}$$

which gives

$$(1-\theta_n+\mu\theta_n)V_{h(n+1)} - \mathcal{N}V_{h(n+1)} \leq 0.$$

So we can say that $(1-\theta_n+\mu\theta_n)V_{h(n+1)}$ is a sub-solution of (4.16) at the step $n+2$. Assumption (S1) implies (4.23), or equivalently

$$V_{n+1} - V_{n+2} \leq \theta_n(1-\mu)V_{n+1}. \tag{4.26}$$

By the inequalities $V_{h0} - V_{h1} \leq V_{h0}$ in $\mathbb{R}^d \times \mathbb{I}$, we obtain $V_{h1} - V_{h2} \leq (1-\mu)V_{h1}$ in $\mathbb{R}^d \times \mathbb{I}$. Then, as in the continuous case,

$$V_{h(n+1)} - V_{h(n+2)} \leq (1-\mu)^{n+1}V_{h(n+1)} \leq (1-\mu)^{n+1}|V_{h0}|_0. \tag{4.27}$$

By (4.18) and (4.19), we can find a function $V_h \in C_{\rm b}(\mathbb{R}^d \times \mathbb{I})$, such that $|V_{hn} - V_h|_0 \to 0$, when $n \to +\infty$. Proposition 2.2 and the stability of solutions imply that V_h is a solution of (3.1). Then, V_{hn} converges to the solution V_h of (3.1), as $n \to +\infty$. Moreover, by (4.19) and since $(1-\mu) < 1$, the following upper bound holds in $\mathbb{R}^d \times \mathbb{I}$ for all $n \geq 0$

$$0 \leq V_{hn} - V_h \leq \sum_{i=n}^{+\infty}(1-\mu)^i|V_{h0}|_0 = \frac{(1-\mu)^n}{1-(1-\mu)}|V_{h0}|_0 = \frac{(1-\mu)^n}{\mu}|V_{h0}|_0. \tag{4.28}$$

□

If (4.14), (4.15) and (4.16) admit solutions V_{hn}, then they converge towards the solutions V_n of (4.1) and (4.2) and we also have (4.18) and (4.20).

5 Lipschitz continuity

We point out that, in order to establish the approximation error of the scheme, we need V to be Lipschitz (or at least Hölder) continuous. In general, the problem (2.9) is expected to have a Hölder continuous solution (see [10]). Assumption (A3) ensures that the jump operator $\mathcal{N}$ is non-expansive in the ∞-norm, and with some additional assumption (including λ large enough) it is possible to prove that the value function is Lipschitz continuous. This claim will be stated precisely and proved in this section.

Lemma 5.1. *Under assumption (A1)–(A5), the viscosity solution V_0 of the HJB equation* (4.1) *is Lipschitz continuous and its Lipschitz constant is given by:*

$$L_{V_0} = \frac{L_\ell}{\lambda - L_f}$$

Proof. This is a classical result and its proof can be found in [1]. □

Now, consider a general HJB equation of the form:

$$\max\Big\{\lambda w(x,q) + H(x,q,D_x w(x,q)), w(x,q) - \Phi(x,q)\Big\} = 0 \quad (x,q) \in \mathbb{R}^d \times \mathbb{I}. \tag{5.1}$$

where $\Phi : \mathbb{R}^d \times \mathbb{I} \to \mathbb{R}$ is Lipschitz continuous. Consider the controlled system :

$$\begin{cases} \dot{X} = f\big(X, q, u(t)\big) \\ X(0) = x, \end{cases} \tag{5.2}$$

where $u \in \mathcal{U}$. Let $\Theta^0 := \mathcal{U} \times \mathbb{R}^+$ be the set of strategies $(u(\cdot), \xi)$ (each pair consists of an admissible control u and a stopping time ξ). By viscosity theory, the value function $w : \mathbb{R}^d \times \mathbb{I}$ defined by:

$$w(x,q) := \inf_{(u,\xi)\in\Theta^0} \left[\int_0^\xi \ell\big(X(t), q, u(t)\big) e^{-\lambda t} dt + e^{-\lambda\xi} \Phi\big(X(\xi), q\big)\right] \tag{5.3}$$

is solution of the equation (5.1). Note that (5.1) is actually a system of q independant equations. Again, by using classical arguments in viscosity theory, we get the following lemma.

Lemma 5.2. *Under assumptions (A1)–(A5), equation* (5.1) *admits a unique bounded Lipschitz continuous viscosity solution w. Moreover, the Lipschitz constant of w satisfies:*

$$L_w = \max\left\{L_\Phi, \frac{L_\ell}{\lambda - L_f}\right\}$$

Proof. The proof can be found in Appendix A.1. □

This Lemma 5.2 and the cascade construction, lead directly to the following conclusion.

Theorem 5.3. *Assume (A1)–(A5). The value function V is Lipschitz continuous and an upper bound of its Lipschitz constant is:*

$$|V|_1 \le \max\{L_{V_0}, L_c\}$$

Proof. Consider the cascade construction and the associated sequence $\{V_n\}$. We claim that for any $n \geq 1$, an upper bound of the Lipschitz constant of V_n is given by:

$$L_{V_n} = \max\{L_{V_0}, L_c\} \tag{5.4}$$

For $n = 0$, this result is stated in Lemma 5.1. Now, assume that (5.4) holds for $n \geq 0$ and let us prove that the statement remains valid for $n + 1$. First, notice that for hypothesis (A5), for every x_1 and x_2

$$\begin{aligned}
\left|\mathcal{N}V_n(x_1, q) - \mathcal{N}V_n(x_2, q)\right| \leq & \Big| \min_{(x',q')\in\mathcal{D}} \big\{V_n(x', q') + c(x_1, q, x', q')\big\} - \\
& - \min_{(x',q')\in\mathcal{D}} \big\{V_n(x', q') + c(x_2, q, x', q')\big\}\Big| \leq \\
& \leq \Big| \sup_{(x',q')\in\mathcal{D}} \big\{c(x_1, q, x', q') - c(x_2, q, x', q')\big\}\Big| \leq \\
& \leq L_c|x_1 - x_2|.
\end{aligned}$$

Hence, by combining the previous inequality with Lemma 5.2, we deduce that

$$|V_{n+1}|_1 \leq \max\left(\frac{L_\ell}{\lambda - L_f}, L_c\right),$$

which coincides with (5.4). Passing to the limit, we conclude the proof. □

6 Error estimates

Before starting the analysis of error estimates for the approximation of (2.9), we first analyze two intermediate problems. The first one corresponds to the first iteration in the cascade problems defined in the previous section.

6.1 The case of the Hamilton-Jacobi equation with obstacles

Consider first the viscosity solution w of the general HJB equation (5.1) and define an approximation w_h of w as solution of the following numerical scheme:

$$\max\Big\{S\big(x, q, w_h(x, q), w_h\big), w_h(x, q) - \Phi(x, q)\Big\} = 0 \quad (x, q) \in \mathbb{R}^d \times \mathbb{I}. \tag{6.1}$$

In order to prove the results in this section, we make the following assumptions. We verify in Appendix A.2 that they hold for the first-order Semi-Lagrangian scheme, but the proof can be extended to other monotone Finite Difference schemes.

(S5) For every sufficiently small $h > 0$ the solution w_h of (6.1) exists.

(S6) The value function w_h is Lipschitz continuous and

$$|w_h|_1 \leq L_{w_h} := \big(1 + (\lambda - L_f)h\big) \max\left\{L_\Phi, \frac{L_\ell}{\lambda - L_f}\right\} \tag{6.2}$$

The proof of the error estimates will use the shaking coefficients and regularization arguments introduced by Krylov in [15, 16]. To use this method, some further notations are needed. Consider a sequence of mollifiers $\{\rho_\varepsilon\}$ defined by:

$$\rho_\varepsilon(x) = \varepsilon^{-d}\rho\left(\frac{x}{\varepsilon}\right), \tag{6.3}$$

where $\rho \in C^\infty(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \rho = 1$, $\operatorname{supp}\{\rho\} \subseteq \bar{B}(0,1)$ and $\rho \geq 0$. The mollification of $\phi \in C_{\mathrm{b}}(\mathbb{R}^d)$ is defined as the convolution:

$$\phi_\varepsilon(x) := \phi * \rho = \int_{\mathbb{R}^d} \phi(x-e)\rho_\varepsilon(e)de. \tag{6.4}$$

If ϕ is Lipschitz continuous, then

$$\big|\phi(x) - \phi_\varepsilon(x)\big| \leq L_\phi \varepsilon, \quad \text{and} \quad \big|D^i\phi_\varepsilon(x)\big| \leq L_\phi \varepsilon^{1-i}\big|\phi|_0. \tag{6.5}$$

Lemma 6.1. *Assume (A1)–(A6). For every sufficiently small $\varepsilon > 0$, the following assertions hold:*

i) There exists a unique solution w^ε of

$$\max\Big\{\lambda w^\varepsilon(x,q) + \max_{|e|\leq\varepsilon} H\big(x+e, q, D_x w^\varepsilon(x,q)\big), w^\varepsilon(x,q) - \Phi(x,q)\Big\} = 0 \quad (x,q) \in \mathbb{R}^d \times \mathbb{I}. \tag{6.6}$$

ii) If w is a solution of (5.1) and $K_\Phi := \frac{L_\ell}{\lambda - L_f} + L_\Phi$, then:

$$|w - w^\varepsilon|_0 \leq \varepsilon K_\Phi,$$

$$|w^\varepsilon|_1 \leq L_w = \max\left\{L_\Phi, \frac{L\ell}{\lambda - L_f}\right\}.$$

iii) Define $w_\varepsilon := w^\varepsilon * \rho_\varepsilon$. Then, there exists $C > 0$ such that w_ε is a classical sub-solution of

$$\max\Big\{\lambda w_\varepsilon(x,q) + H\big(x, q, D_x w_\varepsilon(x,q)\big), u_\varepsilon(x,q) - C\varepsilon - \Phi(x,q)\Big\} \leq 0 \quad (x,q) \in \mathbb{R}^d \times \mathbb{I}. \tag{6.7}$$

Proof. i) The existence and uniqueness of solution w^ε is standard. In particular, concerning existence, w^ε is the value function of the stopping control problem described below (we also report a more general formulation in Appendix A.3).

Consider the following modification of the dynamics described in (2.1):

$$\begin{cases} \dot{X}^\varepsilon(t) = f\big(X^\varepsilon(t) + e(t), Q(t), u(t)\big) \\ X^\varepsilon(0) = x \\ Q(0^+) = q \end{cases} \tag{6.8}$$

where, given $\varepsilon > 0$, $e \in \mathcal{F}^\varepsilon$ with

$$\mathcal{F}^\varepsilon := \big\{e : (0,+\infty) \to \mathbb{R}^d \quad | \quad e \text{ measurable, } |e(t)| \leq \varepsilon \text{ a.e.}\big\}$$

With these dynamics, we define a stopping control problem in which the option to switch between dynamics is replaced by the option to stop at any moment. The stopping time is denoted by ξ and, in case the controller doesn't choose to stop, its value is $+\infty$ by definition.

The control strategy θ^ε consists in the (u, ξ, e), with u is a control input, ξ is the stopping time ξ (which can be finite or infinite) and e is a control function which represent a perturbation in $\mathcal{F}^\varepsilon$. Therefore the set of admissible controls is $\Theta^\varepsilon := \mathcal{U} \times \mathbb{R}^+ \times \mathcal{F}^\varepsilon$.

From [1], we know that the function w^ε defined as

$$w^\varepsilon(x,q) := \inf_{\theta^\varepsilon \in \Theta^\varepsilon} J^\varepsilon(x,q;\theta^\varepsilon) \tag{6.9}$$

where

$$J^\varepsilon(x,q;\theta^\varepsilon) := \int_0^\xi \ell\big(X^\varepsilon(t,q,u) + e(t), q, u(t)\big)e^{-\lambda t}dt + e^{-\lambda\xi}\Phi\big(X^\varepsilon(\xi,q,u), q\big).$$

is the unique solution of (6.6).

ii) The stability result is also proved in Appendix A.3, while the estimate on the Lipschitz constant of w^ε is obtained in Appendix A.1.

iii) First, note that w^ε is sub-solution of the equation:

$$\lambda w^\varepsilon(x,q) + \max_{|e|\le\varepsilon} H(x+e,q,D_x w^\varepsilon(x,q)) \le 0 \quad (x,q)\in\mathbb{R}^d\times\mathbb{I}.$$

By a straightforward adaptation of the arguments in [3, Lemma A3], we prove that w_ε is a sub-solution of

$$\lambda w_\varepsilon(x,q) + H(x,q,D_x w_\varepsilon(x,q)) \le 0 \quad (x,q)\in\mathbb{R}^d\times\mathbb{I}.$$

Moreover, since $w^\varepsilon \le \Phi$ and w^ε and Φ are Lipschitz continuous, for any $x\in\mathbb{R}^d\times\mathbb{I}$ we have:

$$w_\varepsilon(x,q) := \int_{|e|\le 1} w^\varepsilon(x-\varepsilon e,q)\rho(e)de \le \int_{|e|\le 1} w^\varepsilon(x,q)\rho(e)de + L_w\varepsilon \le \Phi(x,q) + L_w\varepsilon.$$

□

The same result holds also for the scheme. Indeed, one can define the perturbed scheme by:

$$\max\Big\{\max_{|e|\le\varepsilon} S\big(x+e,q,w_h^\varepsilon(x,q)\big), w_h^\varepsilon(x,q)-\Phi(x,q)\Big\} = 0 \quad (x,q)\in\mathbb{R}^d\times\mathbb{I}. \tag{6.10}$$

We now need one additional assumption.

(S7) For every sufficiently small $h>0$ the solution w_h^ε of (6.10) exists.

Lemma 6.2. *Assume (S1)–(S7). For $\varepsilon>0$ sufficiently small, define $w_{h,\varepsilon} := w_h^\varepsilon * \rho_\varepsilon$. Then, there exists a constant $C>0$ such that $w_{h,\varepsilon}$ is a classical sub-solution of*

$$\max\Big\{S\big(x,q,w_{h,\varepsilon}(x,q),w_{h,\varepsilon}\big), w_{h,\varepsilon}(x,q)-C\varepsilon-\Phi(x,q)\Big\} \le 0 \quad (x,q)\in\mathbb{R}^d\times\mathbb{I}. \tag{6.11}$$

Proof. This result is derived with the same arguments as in the proof of Lemma 6.1(iii). □

Lastly, we assume that:

(S8) The value function w_h^ε satisfies $|w_h^\varepsilon|_1 \le L_{w_h}$ and

$$|w_h - w_h^\varepsilon|_0 \le \varepsilon K_{w_h,h} \tag{6.12}$$

where w_h is a solution of (6.1) and

$$K_{w_h,h} := \max\left\{(L_\ell + L_{w_h}L_f)h, \frac{L_\ell + L_{w_h}L_f}{\lambda}\right\}.$$

(S9) For an obstacle function $\tilde\Phi$, the solution $\tilde w_h$ of

$$\max\Big\{S\big(x,q,\tilde w_h(x,q),\tilde w_h\big), \tilde w_h(x,q)-\tilde\Phi(x,q)\Big\} = 0 \quad (x,q)\in\mathbb{R}^d\times\mathbb{I}$$

and the solution w_h of (6.1) satisfy

$$\big|w_h(x,q)-\tilde w_h(x,q)\big| \le \big|\Phi(x,q)-\tilde\Phi(x,q)\big| \quad (x,q)\in\mathbb{R}^d\times\mathbb{I}. \tag{6.13}$$

Assumptions (S7) and (S8) are proved in Sections A.2 and A.4 for the case of a monotone Semi-Lagrangian scheme, and can be proved for other classical monotone schemes with the same arguments. Assumption (S9) can be easily checked for monotone SL schemes.

Proposition 6.3. *Assume (A1)–(A5) and (S1)–(S9). If problems* (6.1) *and* (6.10) *admit solutions and* (6.2) *and* (6.12) *hold, then for every* $(x,q) \in \mathbb{R}^d \times \mathbb{I}$, *we have*

$$- \underline{K}_{w_h,h}|h|^\gamma \leq w(x,q) - w_h(x,q) \leq \overline{K}_{w,\Phi}|h|^\gamma \tag{6.14}$$

where

$$\overline{K}_{w,\Phi} := K_\Phi + L_w + K_c L_w |w|_0 |J|$$
$$\underline{K}_{w_h,h} := K_{w_h,h} + L_{w_h} + K_c L_{w_h} |w_h|_0 |J|$$

and

$$\gamma := \min_{i \in J} \frac{k_i}{i} \tag{6.15}$$

according to the definitions in (S3) and Lemma 6.1(ii).

Proof. By Lemma 6.1 (iii), w_ε is a classical sub-solution of (6.7). Therefore, by (S3) and (6.5), we have:

$$\begin{aligned} S\big(h,x,q,w_\varepsilon(x,q),w_\varepsilon\big) &\leq \lambda w_\varepsilon(x,q) + H\big(x,q,Dw_\varepsilon(x,q)\big) + K_\mathrm{c}\mathcal{E}(h,w_\varepsilon) \leq \\ &\leq K_\mathrm{c} \sum_{i\in J} |D^i w_\varepsilon|_0 |h|^{k_i} \leq K_\mathrm{c} \sum_{i\in J} L_w \varepsilon^{1-i} |w|_0 |h|^{k_i} \leq \\ &\leq K_\mathrm{c} L_w |w|_0 \sum_{i\in J} \varepsilon^{1-i}|h|^{k_i}. \end{aligned}$$

By comparison principle of the scheme, we get:

$$w_\varepsilon - w_h \leq K_\mathrm{c} L_w |w|_0 \sum_{i\in J} \varepsilon^{1-i}|h|^{k_i}.$$

In order to determine γ, we substitute $\varepsilon = |h|^\gamma$ in the previous estimate to obtain

$$w_\varepsilon - w_h \leq K_\mathrm{c} L_w |w|_0 \sum_{i\in J} |h|^{\gamma(1-i)+k_i}.$$

So, by choosing $\gamma = \min_{i\in J} \frac{k_i}{i}$, we have

$$w_\varepsilon - w_h \leq K_\mathrm{c} L_w |w|_0 |J| |h|^\gamma.$$

Now, by taking (6.5) and Lemma 6.1(ii) into account, we conclude

$$w - w_h = w - w^\varepsilon + w^\varepsilon - w_\varepsilon + w_\varepsilon - w_h \leq K_\Phi |h|^\gamma + L_w |h|^\gamma + K_\mathrm{c} L_w |w|_0 |J| |h|^\gamma$$

and therefore the upper bound in (6.14) is satisfied.

The lower bound on $w - w_h$ follows with symmetric arguments where a smooth sub-solution of equation (5.1) is constructed from the regularized numerical scheme (6.11). In fact, by Lemma 6.2 we have that $w_{h,\varepsilon}$ is a classical sub-solution of (6.11), and by applying (S3) and (6.5) we obtain

$$\begin{aligned} \lambda w_{h,\varepsilon}(x,q) + H\big(x,q,Dw_{h,\varepsilon}(x,q)\big) &\leq S\big(h,x,q,w_{h,\varepsilon}(x,q),w_{h,\varepsilon}\big) + K_\mathrm{c}\mathcal{E}(h,w_{h,\varepsilon}) \leq \\ &\leq K_\mathrm{c} \sum_{i\in J} |D^i w_{h,\varepsilon}|_0 |h|^{k_i} \leq \\ &\leq K_\mathrm{c} L_{w_h} |w_h|_0 \sum_{i\in J} \varepsilon^{1-i}|h|^{k_i}. \end{aligned}$$

Again, by using the comparison principle and using $\varepsilon = |h|^\gamma$ with $\gamma = \min_{i\in J} \frac{k_i}{i}$, we get

$$w_{h,\varepsilon} - w \le K_c L_{w_h} |w_h|_0 \sum_{i\in J} |h|^{\gamma(1-i)+k_i} \le K_c L_{w_h} |w_h|_0 |J| |h|^\gamma.$$

Now, by taking (6.5), (6.12) and (6.2) into account, we conclude

$$w_h - w = w_h - w_h^\varepsilon + w_h^\varepsilon - w_{h,\varepsilon} + w_{h,\varepsilon} - w \le K_{w_h,h}|h|^\gamma + L_{w_h}|h|^\gamma + K_c L_{w_h}|w_h|_0|J||h|^\gamma$$

and therefore we obtain the lower bound in (6.14). □

6.2 Error estimates for the case without controlled jumps

First, consider the problem (4.1) and its viscosity solution $V_0 \in C_{b,l}(\mathbb{R}^d \times \mathbb{I})$.

Proposition 6.4. *Assume that (A1)–(A5) and (S1)–(S3) hold. Then, if $\lambda > 1$, there exists a constant $C_0 > 0$ such that*

$$\big|V_0(x,q) - V_{h0}(x,q)\big| \le C_0|h|^\gamma \quad (x,q) \in \mathbb{R}^d \times \mathbb{I}$$

where, in our case

$$C_0 := 2K_c|J| \max\{L_{V_0}|V_0|_0, L_{V_{h0}}|V_{h0}|_0\}$$

and $\gamma := \min_{i\in J} \frac{k_i}{i}$, according to the definitions in (S3) and Lemma 6.1(ii).

Proof. This is a classical result, proved in [9]. □

6.3 The error estimate for the problem with n switches

First, for every sufficiently small $\varepsilon > 0$, we define V_n^ε as the viscosity solution of

$$\begin{aligned} \max\Big\{\lambda V_n(x,q) + \max_{|e|\le\varepsilon} H\big(x+e, q, D_x V_n(x,q)\big), \\ V_n(x,q) - \mathcal{N} V_{n-1}(x,q)\Big\} = 0 \end{aligned} \qquad (x,q) \in \mathbb{R}^d \times \mathbb{I}. \tag{6.16}$$

We recall that (6.16) has a unique solution by Lemma 6.1 (i).

Lemma 6.5. *Let V_n^ε be the viscosity solution of* (6.16)*, for $n \ge 1$. Then, an upper bound of the Lipschitz constant of V_n^ε is*

$$|V_n^\varepsilon|_1 \le \max\{L_{V_0}, L_c\}. \tag{6.17}$$

Proof. Using the same methods as for sequence (4.3), we can show that

$$0 \le \cdots \le V_n^\varepsilon \le \cdots \le V_2^\varepsilon \le V_1^\varepsilon \le V_0^\varepsilon \tag{6.18}$$

Combining with (6.17), get

$$0 \le \cdots \le L_{V_n^\varepsilon} \le \cdots \le L_{V_2^\varepsilon} \le L_{V_1^\varepsilon} \le \max\{L_{V_0}, L_c\} \tag{6.19}$$

□

We can give now the error estimate of the upper and lower bound of the difference between V_n and V_{hn}. We recall that C_0 has been defined in Proposition 6.4.

Proposition 6.6. *For $n \geq 1$, let $V_n \in C_{b,l}(\mathbb{R}^d \times \mathbb{I})$ be the unique viscosity solution of (4.2), and $V_{hn} \in C_b(\mathbb{R}^d \times \mathbb{I})$ the unique solution of (4.16). Then, on $\mathbb{R}^d \times \mathbb{I}$ we have*

$$-\underline{C}_n|h|^\gamma \leq V_n(x,q) - V_{hn}(x,q) \leq \overline{C}_n|h|^\gamma \tag{6.20}$$

where, for every $n \geq 1$, there exist positive constants $\overline{K}_{V_{n-1}}$ and $\underline{K}_{V_{h(n-1)},h}$ such that

$$\begin{aligned} \overline{C}_n &:= \overline{C}_{n-1} + \overline{K}_{V_{n-1}} \\ \underline{C}_n &:= \underline{C}_{n-1} + \underline{K}_{V_{h(n-1)},h}. \end{aligned} \tag{6.21}$$

Proof. . We prove the proposition by induction over n, starting from the upper bound.

Let $n = 1$. We want to estimate the difference

$$V_1(x,q) - V_{h1}(x,q) = V_1(x,q) - \tilde{V}_{h1}(x,q) + \tilde{V}_{h1}(x,q) - V_{h1}(x,q)$$

where $\tilde{V}_{h1}$ is the solution of

$$\max\Big\{S\big(x,q,\tilde{V}_{h1}(x,q),\tilde{V}_{h1}\big), \tilde{V}_{h1}(x,q) - \mathcal{N}V_0(x,q)\Big\} = 0 \quad (x,q) \in \mathbb{R}^d \times \mathbb{I}.$$

By applying Proposition 6.3, (6.13) and Proposition 6.4 we obtain

$$V_1(x,q) - V_{h1}(x,q) \leq \overline{K}_{V_1,\mathcal{N}V_0}|h|^\gamma + C_0|h|^\gamma.$$

Note that, for every $n \geq 1$, the constant $\overline{K}_{V_n,\mathcal{N}V_{n-1}}$ coincides with the constant $\overline{K}_{w,\Phi}$ defined in Proposition 6.3 in the case $w = V_n$ and $\Phi = \mathcal{N}V_{n-1}$ and, by assumption (A3), it can be simplified to

$$\begin{aligned} \overline{K}_{V_{n-1}} &:= \frac{L_\ell}{\lambda - L_f} + L_{V_{n-1}}\frac{L_f}{\lambda} + L_{V_{n-1}} + K_c L_{V_{n-1}}|V_{n-1}|_0|J| \geq \\ &\geq \frac{L_\ell}{\lambda - L_f} + L_{V_{n-1}}\frac{L_f}{\lambda} + L_{V_n} + K_c L_{V_n}|V_n|_0|J| \geq \\ &\geq \frac{L_\ell}{\lambda - L_f} + L_{\mathcal{N}V_{n-1}}\frac{L_f}{\lambda} + L_{V_n} + K_c L_{V_n}|V_n|_0|J| =: \overline{K}_{V_n,\mathcal{N}V_{n-1}}. \end{aligned} \tag{6.22}$$

We also recall that

$$C_0 := 2K_c|J|\max\{L_{V_0}|V_0|_0, L_{V_{h0}}|V_{h0}|_0\}$$

while K_c and J are defined in the consistency hypothesis (S3).

By the definition of $\overline{K}_{V_0}$ in (6.22), we obtain

$$V_1(x,q) - V_{h1}(x,q) \leq \overline{K}_{V_0}|h|^\gamma + C_0|h|^\gamma$$

and, defining $\overline{C}_1 := \overline{K}_{V_0} + C_0$, we have

$$V_1(x,q) - V_{h1}(x,q) \leq \overline{C}_1|h|^\gamma \quad (x,q) \in \mathbb{R}^d \times \mathbb{I}.$$

Assume now that the result is true at the step n. For the step $n+1$, applying Proposition 6.3 and (6.13), we get

$$\begin{aligned} V_{n+1}(x,q) - V_{h(n+1)}(x,q) &= V_{n+1}(x,q) - \tilde{V}_{h(n+1)}(x,q) + \tilde{V}_{h(n+1)}(x,q) - V_{h(n+1)}(x,q) \leq \\ &\leq \overline{K}_{V_n}|h|^\gamma + \big|V_n(x,q) - V_{hn}(x,q)\big| \leq \\ &\leq \overline{K}_{V_n}|h|^\gamma + \overline{C}_n|h|^\gamma. \end{aligned}$$

Hence, by taking $\overline{C}_{n+1} := \overline{K}_{V_n} + \overline{C}_n$ we finally obtain

$$V_{n+1}(x,q) - V_{h(n+1)}(x,q) \le \overline{C}_{n+1}|h|^\gamma.$$

For the lower bound, the base case of the induction can be obtained in a similar way by applying Proposition 6.3 and (6.13):

$$\begin{aligned} V_{h1}(x,q) - V_1(x,q) = V_{h1}(x,q) - \tilde{V}_{h1}(x,q) + \tilde{V}_{h1}(x,q) - V_1(x,q) \le \\ \le C_0|h|^\gamma + \underline{K}_{V_{h0},h}|h|^\gamma = \underline{C}_1|h|^\gamma, \end{aligned}$$

in which $\underline{C}_1 := \underline{K}_{V_{h0},h} + C_0$, and, for every $n \ge 1$, the constant $\underline{K}_{V_{hn},h}$ coincides with $\underline{K}_{w_h,h}$ defined in Proposition 6.3 in the case $w_h = V_{hn}$:

$$\underline{K}_{V_{hn},h} := \max\left\{(L_\ell + L_{V_{hn}}L_f)h, \frac{L_\ell + L_{V_{hn}}L_f}{\lambda}\right\} + L_{V_{hn}} + K_{\mathrm{c}}L_{V_{hn}}|V_{hn}|_0|J|.$$

The rest of the induction follows the same steps of the previous case, leading to

$$V_{h(n+1)}(x,q) - V_{n+1}(x,q) \le \underline{C}_{n+1}|h|^\gamma$$

with $\underline{C}_{n+1} := \underline{K}_{V_{hn},h} + \underline{C}_n$. □

6.4 The error estimate for $V - V_h$

Before stating our main result, we define

$$\overline{D}_{n-1} := \overline{C}_n - \overline{C}_{n-1} = \overline{K}_{V_{n-1}}.$$

where $\overline{C}_n$ has been defined in (6.21). The definition of K_Φ in Lemma 6.1 (ii) and (6.19) imply that $\overline{D}_n \le \overline{D}_0$, and hence

$$\overline{C}_n \le C_0 + n\overline{D}_0. \tag{6.23}$$

Similarly, if we define

$$\underline{D}_{n-1} := \underline{C}_n - \underline{C}_{n-1} = \underline{K}_{V_{h(n-1)},h}$$

from the definition of $K_{w_h,h}$ in (6.12) and (4.17) we have that $\underline{D}_n \le \underline{D}_0$, and hence:

$$\underline{C}_n \le C_0 + n\underline{D}_0. \tag{6.24}$$

Theorem 6.7. *Assume (A1)–(A5) and (S1)–(S9). Let $V \in C_{b,l}(\mathbb{R}^d)$ be the unique viscosity solution of (2.9), and $V_h \in C_b(\mathbb{R}^d)$ the unique solution of (4.14). Then there exist $\overline{C} > 0$ and $\underline{C} > 0$ such that, for h small enough,*

$$-\underline{C}\left|\ln|h|\right||h|^\gamma \le V(x,q) - V_h(x,q) \le \overline{C}\left|\ln|h|\right||h|^\gamma \quad (x,q) \in \mathbb{R}^d \times \mathbb{I} \tag{6.25}$$

Proof. We start with the upper bound. By (4.5), (6.23) and (4.20) we obtain the following estimate

$$V - V_h = V - V_n + V_n - V_{hn} + V_{hn} - V_h \le \frac{(1-\mu)^n}{\mu}|V_0|_0 + (C_0 + n\overline{D}_0)|h|^\gamma + \frac{(1-\mu)^n}{\mu}|V_{h0}|_0$$

which can be rearranged as

$$V - V_h \le \frac{|V_0|_0 + |V_{h0}|_0}{\mu}(1-\mu)^n + \overline{D}_0|h|^\gamma n + C_0|h|^\gamma.$$

For the lower bound, by the same arguments and using (6.24) instead of (6.23), we have

$$V_h - V \leq \frac{(1-\mu)^n}{\mu}|V_{h0}|_0 + (C_0 + n\underline{D}_0)|h|^\gamma + \frac{(1-\mu)^n}{\mu}|V_0|_0$$

or, equivalently

$$V_h - V \leq \frac{|V_0|_0 + |V_{h0}|_0}{\mu}(1-\mu)^n + \underline{D}_0|h|^\gamma n + C_0|h|^\gamma.$$

The idea now is to minimize with respect to n the estimates on the upper and lower bound:

$$\overline{E}(n) := a(1-\mu)^n + \overline{b}n + c$$
$$\underline{E}(n) := a(1-\mu)^n + \underline{b}n + c$$

where $a := \frac{|V_0|_0+|V_{h0}|_0}{\mu}$, $\overline{b} := \overline{D}_0|h|^\gamma$, $\underline{b} := \underline{D}_0|h|^\gamma$ and $c := C_0|h|^\gamma$.

By a straightforward application of [6, Lemma 6.1] to $\overline{E}(n)$ we have

$$\begin{cases} V - V_h \leq -\frac{\overline{b}}{\ln(1-\mu)} + c & -\frac{\overline{b}}{a\ln(1-\mu)} \geq 1 \\ V - V_h \leq -\frac{(1-\mu)\overline{b}}{\ln(1-\mu)} + \overline{b}\left(\log_{1-\mu}\left(-\frac{\overline{b}}{a\ln(1-\mu)}\right) + 1\right) + c & \text{else.} \end{cases}$$

More explicitly, if

$$-\frac{\overline{D}_0|h|^\gamma}{\frac{|V_0|_0+|V_{h0}|_0}{\mu}\ln(1-\mu)} \geq 1,$$

then the upper bound for $V - V_h$ is

$$\left(-\frac{\overline{D}_0}{\ln(1-\mu)} + C_0\right)|h|^\gamma,$$

otherwise

$$\left(-\frac{(1-\mu)\overline{D}_0}{\ln(1-\mu)} + \overline{D}_0\left(\log_{1-\mu}\left(-\frac{\mu\overline{D}_0|h|^\gamma}{(|V_0|_0 + |V_{h0}|_0)\ln(1-\mu)}\right) + 1\right) + C_0\right)|h|^\gamma.$$

In this second case, the factor multiplying $|h|^\gamma$ is $O(\ln|h|) + O(1)$.

The same can be proven for the lower bound, replacing $\overline{D}_0$ with $\underline{D}_0$, thus proving the result. □

A Appendix

A.1 The upper bounds of Lipschitz constants

Proof of Lemma 5.2. For $q \in \mathbb{I}$ and $\epsilon > 0$, set

$$m_\epsilon := \sup_{x,y\in\mathbb{R}^d} \varphi(x,y) := \sup_{x,y\in\mathbb{R}^d}\left\{w(x,q) - w(y,q) - \frac{\delta}{2}|x-y|^2 - \frac{\epsilon}{2}(|x|^2+|y|^2)\right\}.$$

Let $x_0, y_0 \in \mathbb{R}^d$ such that $m_\epsilon = \varphi(x_0,y_0)$. Taking into account the HJB equation satisfied by w and applying the notion of viscosity solution, we get:

$$0 \leq \max\big\{\lambda w(y_0,q) + H(y_0,q,p_y) - \lambda w(x_0,q) - H(x_0,q,p_x),$$
$$w(y_0,q) - \Phi(y_0,q) - w(x_0,q) + \Phi(x_0,q)\big\}.$$

Two cases have to be considered.

1. The max is attained by its first argument.

   In this case, we get

$$\lambda w(y_0, q) + H(y_0, q, p_y) - \lambda w(x_0, q) - H(x_0, q, p_x) \geq 0$$

   where

$$p_x = \delta(x_0 - y_0) + \epsilon x_0$$
$$p_y = \delta(x_0 - y_0) - \epsilon y_0.$$

   This is a standard case (see [1]), and we have that

$$w(x, q) - w(y, q) \leq \frac{L_\ell}{\lambda - L_f}|x - y|.$$

2. The max is attained by its second argument.

   In this case,

$$w(y_0, q) - \Phi(y_0, q) - w(x_0, q) + \Phi(x_0, q) \geq 0,$$

   so that we get $w(x_0, q) - w(y_0, q) \leq L_\Phi |x_0 - y_0|$, and we can infer that

$$m_\epsilon \leq L_\Phi |x_0 - y_0| - \frac{\delta}{2}|x_0 - y_0|^2. \tag{A.1}$$

   Setting $r := |x_0 - y_0|$, we have $\max_{r \geq 0} \left\{ L_\Phi r - \frac{\delta}{2} r^2 \right\} = L_\Phi^2 / 2\delta$, and hence we obtain

$$m_\epsilon \leq \frac{L_\Phi^2}{2\delta}. \tag{A.2}$$

   Define now $m := \lim_{\epsilon \to 0} m_\epsilon$. Applying a simple calculus argument (see [14, Lemma 2.3]), for fixed δ, we have:

$$m = \sup_{x, y \in \mathbb{R}^d} \left\{ w(x, q) - w(y, q) - \delta |x - y|^2 \right\} \leq \frac{L_\Phi^2}{2\delta},$$

   where the inequality follows from (A.2). Therefore, by definition of m, we have that:

$$w(x, q) - w(y, q) \leq \frac{L_\Phi^2}{2\delta} + \frac{\delta}{2}|x - y|^2.$$

   Observing now that

$$\min_{\delta \geq 0} \left\{ \frac{L_\Phi^2}{2\delta} + \frac{\delta}{2}|x - y|^2 \right\} = L_\Phi |x - y|,$$

   we finally obtain:

$$w(x, q) - w(y, q) \leq L_\Phi |x - y|.$$

In conclusion, for both cases, we have

$$L_w = \max \left\{ L_\Phi, \frac{L_\ell}{\lambda - L_f} \right\},$$

and, using similar arguments, we can also bound L_{w^ϵ} as:

$$L_{w^\epsilon} = \max \left\{ L_\Phi, \frac{L_\ell}{\lambda - L_f} \right\}.$$

□

A.2 Lipschitz stability for the SL scheme

In this section, we prove that the numerical approximation w_h for the solution w of the obstacle problem is Lipschitz continuous. We consider schemes approximating (5.1) in the fixed point form:

$$W_h(x,q) = \min\left\{\Sigma^h(x,q,W_h), \Phi(x,q)\right\} \quad (x,q) \in \mathbb{R}^d \times \mathbb{I}, \tag{A.3}$$

where, in the case of Semi-Lagrangian schemes, the operator Σ^h reads

$$\Sigma^h(x,q,W_h) := \Pi_{\Delta x} \circ \min_{u\in U}\Big\{h\ell(x,q,u) + e^{-\lambda h} W_h\big(x + hf(x,q,u), q\big)\Big\}, \tag{A.4}$$

where $\Pi_{\Delta x}$ is an interpolation operator based on a space grid of step Δx. It is well-known that Σ^h is non-expansive in the ∞-norm, provided the interpolation operator $\Pi_{\Delta x}$ is monotone (see [11]).

Theorem A.1. *Under assumptions (A1)–(A6), (S1)–(S4), the solution W_h of problem* (6.1) *obtained with the Semi-Lagrangian scheme* (A.3)–(A.4) *is Lipschitz continuous with*

$$|W_h|_1 \le L_{W_h} = \big(1 + (\lambda - L_f)h\big) \max\left\{L_\Phi, \frac{L_\ell}{\lambda - L_f}\right\}.$$

Proof. For $(x,q) \in \mathbb{R}^d \times \mathbb{I}$, consider the iterative solution of the fixed point equation (A.3):

$$W_h^{(k+1)}(x,q) = \min\Big\{\Sigma^h\big(x,q,W_h^{(k)}\big), \Phi(x,q)\Big\}$$

where $W_h^{(k)}$ is the approximation of W_h at the k–th iteration. For any $x_1, x_2 \in \mathbb{R}^d$, we have

$$\begin{aligned}\big|W_h^{(k+1)}(x_1,q) - W_h^{(k+1)}(x_2,q)\big| &\le \max\Big\{hL_\ell + e^{-\lambda h}(1+\lambda L_f)L_{W_h^{(k)}}, L_\Phi\Big\} \le \\ &\le \max\Big\{hL_\ell + e^{-(\lambda - L_f)h}L_{W_h^{(k)}}, L_\Phi\Big\} \le \\ &\le \max\left\{\frac{L_\ell}{\lambda - L_f}, L_\Phi\right\} \max\left\{(\lambda - L_f)h + \frac{e^{-(\lambda-L_f)h}L_{W_h^{(k)}}}{\max\left\{\frac{L_\ell}{\lambda - L_f}, L_\Phi\right\}}, 1\right\}.\end{aligned}$$

By setting

$$m := (\lambda - L_f)h > 0,$$
$$M_k := \frac{L_{W_h^{(k)}}}{\max\left\{\frac{L_\ell}{\lambda-L_f}, L_\Phi\right\}},$$

we have

$$M_{k+1} \le \max\{m + e^{-m}M_k, 1\}.$$

Note that, if $M_k \le 1 + m$, then $e^{-m}M_k \le e^{-m}(1+m) \le 1$. Hence,

$$M_{k+1} \le \max\{m + e^{-m}M_k, 1\} \le \max\{1+m, 1\} = 1 + m.$$

It suffices therefore to initialize the fixed point iterations with $W_h^{(0)}$ such that $M_0 = 0$ to guarantee $M_k \le 1 + m$ for every $k \ge 0$, and, by the definitions of m and M_k, we obtain

$$\frac{L_{W_h^{(k)}}}{\max\left\{\frac{L_\ell}{\lambda-L_f}, L_\Phi\right\}} \le 1 + (\lambda - L_f)h$$

which implies

$$L_{W_h^{(k)}} \le \big(1 + (\lambda - L_f)h\big) \max\left\{\frac{L_\ell}{\lambda - L_f}, L_\Phi\right\}.$$

Now, since $W_h^{(k)}$ converges towards the solution W_h of the scheme (A.3)–(A.4) as $k \to +\infty$, we conclude

$$L_{W_h} \le \big(1 + (\lambda - L_f)h\big) \max\left\{\frac{L_\ell}{\lambda - L_f}, L_\Phi\right\}.$$

□

A.3 Estimate on the perturbed value function of the stopping problem

The cost function corresponding to the stopping problem can be defined as :

$$J(x, q; \theta) := \int_0^\xi \ell\big(X(t, q, u), q, u(t)\big)e^{-\lambda t}dt + e^{-\lambda\xi}\Phi\big(X(\xi, q, u), q\big)$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. Then, the value function given by:

$$w(x, q) := \inf_{\theta\in\Theta^0} J(x, q; \theta),$$

satisfies (5.1). Now, for a given ε, we replace the dynamics (2.1) with (6.8), and define the value function w^ε:

$$w^\varepsilon(x, q) := \inf_{\theta^\varepsilon\in\Theta^\varepsilon} J^\varepsilon(x, q; \theta^\varepsilon)$$

with $\Theta^\varepsilon := \mathcal{U} \times \mathbb{R}_+ \times \mathcal{F}^\varepsilon$ and

$$J^\varepsilon(x, q; \theta^\varepsilon) := \int_0^\xi \ell\big(X^\varepsilon(t, q, u) + e(t), q, u(t)\big)e^{-\lambda t}dt + e^{-\lambda\xi}\Phi\big(X^\varepsilon(\xi, q, u), q\big).$$

With this definition, the value function w^ε is the unique solution of the equation:

$$\max\Big(\lambda w^\varepsilon(x, q) + \max_{|e|\le\varepsilon} H\big(x + e, q, D_x w^\varepsilon(x, q)\big), w^\varepsilon(x, q) - \Phi(x, q)\Big) = 0 \quad \text{in } \mathbb{R}^d \times \mathbb{I}, \tag{A.5}$$

here, we have used the notation $\cdot^\varepsilon$ to distinguish between the quantities related to the perturbed problems and the ones related to the unperturbed problem. Note that the equations in the system (A.5) (reps. (A.5)) are not connected. So in the sequel, we will drop the dependency with respect to q.

The rest of this section is dedicated to deriving an estimate for the difference between w^ε and w. Since every control $\theta \in \Theta^0$ can be considered as an admissible control in Θ^ε with a perturbation function $e \equiv 0$. Hence, we have:

$$w^\varepsilon(x) \le w(x) \quad x \in \mathbb{R}^d. \tag{A.6}$$

Let $\theta = (u, \xi) \in \Theta^0$ and let $e \in \mathcal{F}^\varepsilon$ for some $\varepsilon > 0$. Let X be a trajectory solution of (2.1) associated to θ and let X^ε solution of (6.8) associated to $\theta^\varepsilon = (u, \xi, e)$. For every $t > 0$, we have:

$$\begin{aligned}
|X_x^\varepsilon(t) - X_x(t)| &\le \int_0^t \big|f\big(X_x^\varepsilon(s) + e(s), u(s)\big) - f\big(X_x(s), u(s)\big)\big|\, ds \\
&\le \int_0^t \big(\varepsilon L_f + L_f\big|X_x^\varepsilon(s) - X_x(s)\big|\big)\, ds.
\end{aligned}$$

Then applying Grönwall's inequality we obtain

$$\big|X_x^\varepsilon(t) - X_x(t)\big| \le \varepsilon\big(e^{L_f t} - 1\big) \quad \text{for every } t \ge 0. \tag{A.7}$$

We can now derive the estimate error between the solutions of problems (6.6) and (5.1).

Theorem A.2. *Assume (A1)–(A6) hold. Then, for every $x \in \mathbb{R}^d$ and $\varepsilon > 0$ we have:*

$$\big|w(x,q) - w^\varepsilon(x,q)\big| \le \varepsilon\left(\frac{L_\ell}{\lambda - L_f} + L_\Phi\right).$$

Proof. From the definition of the value function w^ε we know that for each $\delta > 0$ there exists $\theta^\varepsilon_\delta = (u_\delta, \xi_\delta, e_\delta) \in \Theta^\varepsilon$ such that

$$J^\varepsilon(x,q;\theta^\varepsilon_\delta) \le w^\varepsilon(x,q) + \delta.$$

Set $\theta_\delta = (u_\delta, \xi_\delta)$ and denote by $X^\varepsilon_{\delta,x}$ (resp. $X_{\delta,x}$) the solution of (6.8) (resp. of (2.1)) associated to u_δ. It follows that

$$\begin{aligned}
0 \le w(x,q) - w^\varepsilon(x,q) &\le J(x,q;\theta_\delta) - J^\varepsilon(x,q;\theta^\varepsilon_\delta) + \delta\\
&\le \int_0^{\xi_\delta} e^{-\lambda s}|\ell(X_{\delta,x}(s)+e(s),u_\delta(s)) - \ell(X^\varepsilon_{\delta,x}(s),u_\delta(s))|\,ds\\
&\quad + e^{-\lambda\xi_\delta}|\Phi(X^\varepsilon_{\delta,x}(T)) - \Phi(X_{\delta,x}(T))| + \delta\\
&\le L_\ell\int_0^{\xi_\delta} e^{-\lambda s}\Big[|X^\varepsilon_{\delta,x}(s) - X_{\delta,x}(s)| + |e(s)|\Big]\,ds\\
&\quad + e^{-\lambda\xi_\delta}L_\Phi|X^\varepsilon_{\delta,x}(\xi_\delta) - X_{\delta,x}(\xi_\delta)| + \delta\\
&\le \varepsilon L_\ell\int_0^{\xi_\delta} e^{-(\lambda-L_f)s}\,ds + \varepsilon L_\Phi e^{-\lambda\xi_\delta}(e^{L_f\xi_\delta} - 1) + \delta\\
&\le \varepsilon\left(L_\ell\frac{1-e^{-(\lambda-L_f)\xi_\delta}}{\lambda - L_f} + L_\Phi e^{-(\lambda-L_f)\xi_\delta}\right) + \delta\\
&\le \varepsilon\left(\frac{L_\ell}{\lambda - L_f} + L_\Phi\right) + \delta.
\end{aligned}$$

The above estimate being valid for any $\delta > 0$, we can conclude that the statement of the theorem is proved. ☐

A.4 Estimate on the perturbed numerical approximation

We want to examine here the difference between the numerical approximations of respectively the QVI with a constant obstacle and its perturbed version in the case of a Semi-Lagrangian scheme.

We recall that the unperturbed system is

$$\max\Big\{\lambda w(x,q) + H\big(x,q,D_x w(x,q)\big), w(x,q) - \Phi(x,q)\Big\} \le 0 \qquad (x,q) \in \mathbb{R}^d \times \mathbb{I}. \tag{A.8}$$

It can be approximated with the scheme

$$W_h(x,q) = \mathcal{T}^h(x,q,W_h) := \min\big\{\Sigma^h(x,q,W_h), \Phi(x,q)\big\}(x,q) \in (\mathbb{R}^d \times \mathbb{I}). \tag{A.9}$$

The perturbed SL scheme is obtained by replacing Σ^h in (A.9) with the mapping

$$\Sigma^{\varepsilon,h}(x,q,W^\varepsilon_h) = \Pi_{\Delta x} \circ \min_{u\in U, |e|\le\varepsilon}\Big\{h\ell(x+e,q,u) + (1-\lambda h)W^\varepsilon_h\big(x + hf(x+e,q,u),q\big)\Big\}. \tag{A.10}$$

We start by giving the following general result:

Theorem A.3. *Let (A1)–(A6) and (S1)–(S9) hold, and let W_h and W_h^ε be respectively solution of* (A.9) *and its perturbed version* (A.10) *with Φ finite or infinite. Then, the perturbed SL scheme has a unique bounded and uniformly Lipschitz continuous solution W_h^ε.*

Proof. It suffices to note that, with the addition of the term e, the problem still satisfies the basic assumptions, and all the relevant constants of the problem remain unchanged. Then, the result follows from Theorem A.1, implying

$$|W_h^\varepsilon|_1 \le \big(1 + (\lambda - L_f)h\big) \max\left\{L_\Phi, \frac{L_\ell}{\lambda - L_f}\right\}.$$

□

Let now W_h^ε denote the numerical solution for the perturbed SL scheme. We prove the following.

Theorem A.4. *Let (A1)–(A6) and (S1)–(S9) hold, and let W_h and W_h^ε be respectively solution of* (A.9) *and its perturbed version* (A.10) *with Φ finite or infinite. Then, for ε and h small enough, we have*

$$|W_h - W_h^\varepsilon|_0 \le \varepsilon K_{W_h,h} \tag{A.11}$$

with

$$K_{W_h,h} := \max\left\{(L_\ell + L_{W_h} L_f)h, \frac{L_\ell + L_{W_h} L_f}{\lambda}\right\}.$$

Proof. We recall that both the exact and the approximate solutions for either the original or the perturbed problem are Lipschitz continuous.

Using a scheme in fixed point SL form, the unperturbed QVI is approximated by (A.9), whereas its perturbed version is given by

$$W_h^\varepsilon(x,q) = \mathcal{T}^{\varepsilon,h}(x,q,W_h^\varepsilon) := \min\left\{\Sigma^{\varepsilon,h}(x,q,W_h^\varepsilon), \Phi(x,q)\right\} \qquad (x,q) \in (\mathbb{R}^d \times \mathbb{I}). \tag{A.12}$$

The plan is to apply the two schemes to Lipschitz continuous numerical solutions W_h and W_h^ε and estimate, for the various operators, differences of the form

$$\begin{aligned} \big|T^h(\,\cdot\,,\,\cdot\,,W_h) - T^{\varepsilon,h}(\,\cdot\,,\,\cdot\,,W_h^\varepsilon)\big|_0 \le \big|T^h(\,\cdot\,,\,\cdot\,,W_h) - T^h(\,\cdot\,,\,\cdot\,,W_h^\varepsilon)\big|_0 + \\ + \big|T^h(\,\cdot\,,\,\cdot\,,W_h^\varepsilon) - T^{\varepsilon,h}(\,\cdot\,,\,\cdot\,,W_h^\varepsilon)\big|_0 \end{aligned} \tag{A.13}$$

Using now, for $T = \mathcal{T}^h, \mathcal{T}^{\varepsilon,h}, \Sigma^h, \Sigma^{\varepsilon,h}$ and $U = W_h, W_h^\varepsilon$, the shorthand notation

$$T(U) := T(\,\cdot\,,\cdot\,,U)$$

we can single out three cases:

a) $\mathcal{T}^h(x,q,W_h) = \Sigma^h(x,q,W_h)$ and $\mathcal{T}^{\varepsilon,h}(x,q,W_h^\varepsilon) = \Sigma^{\varepsilon,h}(x,q,W_h^\varepsilon)$.

In this case, we can bound the first term in the right-hand side of (A.13) as

$$\big|\mathcal{T}^h(x,q,W_h) - \mathcal{T}^h(x,q,W_h^\varepsilon)\big|_0 = \big|\Sigma^h(x,q,W_h) - \Sigma^h(x,q,W_h^\varepsilon)\big|_0 \le (1-\lambda h)\big|W_h - W_h^\varepsilon\big|_0, \tag{A.14}$$

which is a known property of the SL scheme. For the second, considering the Lipschitz continuity of ℓ and f and the bound on $|e|$, we have

$$\begin{aligned} \big|\ell(x,q,u) - \ell(x+e,q,u)\big| &\le L_\ell \varepsilon, \\ \big|f(x,q,u) - f(x+e,q,u)\big| &\le L_f \varepsilon \end{aligned}$$

so that, taking into account the Lipschitz continuity of W_h, by a standard argument we obtain

$$\big|\Sigma^h(x,q,W_h) - \Sigma^{\varepsilon,h}(x,q,W_h)\big|_0 \le (L_\ell + L_{W_h} L_f)h\varepsilon. \tag{A.15}$$

b) $\mathcal{T}^h(x,q,W_h) = \Phi(x,q) = \mathcal{T}^{\varepsilon,h}(x,q,W_h^\varepsilon)$. In this case there is nothing else to prove.

c) The min is achieved by different operators, e.g., let $\mathcal{T}^h(x,q,W_h) = \Sigma^h(x,q,W_h)$ and $\mathcal{T}^h(x,q,W_h^\varepsilon) = \Phi(x,q)$. Working in terms of unilateral estimates, we have

$$\mathcal{T}^h(x,q,W_h) - \mathcal{T}^h(x,q,W_h^\varepsilon) = \Sigma^h(x,q,W_h) - \Phi(x,q) \le \Phi(x,q) - \Phi(x,q) = 0$$

in which we get the inequality by replacing the argmin in $\Theta^h(x,q,W_h)$ with the other choice. In a parallel form, we obtain the reverse inequality, as

$$\begin{aligned} \mathcal{T}^h(x,q,W_h^\varepsilon) - \mathcal{T}^h(x,q,W_h) = \Phi(x,q) - \Sigma^h(x,q,W_h) &\le \\ \le \Sigma^h(x,q,W_h^\varepsilon) - \Sigma^h(x,q,W_h) &\le \\ \le (1-\lambda h)|W_h - W_h^\varepsilon|_0 & \end{aligned}$$

The same arguments can then be applied to the case in which the choice of the operators is reversed, so that we finally obtain (A.14).

We obtain therefore, by iterating the estimate (A.13) in (A.9) and (A.12) from the same initial guess $W_h^{(0)} = W_h^{\varepsilon(0)}$:

$$|W_h - W_h^\varepsilon|_0 \le (L_\ell + L_{W_h} L_f)\varepsilon h \sum_{k\ge 0}(1-\lambda h)^k = \frac{L_\ell + L_{W_h} L_f}{\lambda}\varepsilon$$

We can therefore conclude by collecting all the cases above in the bound

$$|W_h - W_h^\varepsilon|_0 \le \max\left\{(L_\ell + L_{W_h}L_f)h, \frac{L_\ell + L_{W_h}L_f}{\lambda}\right\}\varepsilon.$$

□

References

[1] M. Bardi and I. Capuzzo Dolcetta. *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*. Birkhuser Mathematics. Birkhauser, 1997.

[2] G. Barles, S. Dharmatti, and M. Ramaswamy. Unbounded viscosity solutions of hybrid control systems. *ESAIM: Control, Optimisation and Calculus of Variations*, 16(1):176–193, 2010.

[3] G. Barles and E. R. Jakobsen. On the convergence rate of approximation schemes for Hamilton–Jacobi–Bellman equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 36(1):33–54, 2002.

[4] G. Barles and P. E. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Analysis*, 4(3):271–283, 1991.

[5] A. Bensoussan and J. L. Menaldi. Hybrid control and dynamic programming. *Dynamics of Continuous, Discrete and Impulsive Systems - Series B: Application and Algorithm*, 3(4):395–442, 1997.

[6] J. F. Bonnans, S. Maroso, and H. Zidani. Error estimates for a stochastic impulse control problem. *Applied Mathematics and Optimization*, 55(3):327357, 2007.

[7] M. S. Branicky, V. S. Borkar, and S. K. Mitter. A unified framework for hybrid control: model and optimal control theory. *IEEE Transactions on Automatic Control*, 43(1):31–45, 1998.

[8] F. Camilli and M. Falcone. An approximation scheme for the optimal control of diffusion processes. *ESAIM: Mathematical Modelling and Numerical Analysis*, 29(1):97–122, 1995.

[9] I. Capuzzo Dolcetta and H. Ishii. Approximate solutions of the bellman equation of deterministic control theory. *Applied Mathematics and Optimization*, 11(1):161–181, 1984.

[10] S. Dharmatti and M. Ramaswamy. Hybrid control systems and viscosity solutions. *SIAM Journal on Contol and Optimization*, 44(1):1259–1288, 2005.

[11] M. Falcone and R. Ferretti. *Semi-Lagrangian approximation schemes for linear and Hamilton-Jacobi equations*, volume 133 of *Other Titles in Applied Mathematics*. SIAM, 2014.

[12] R. Ferretti and H. Zidani. Monotone numerical schemes and feedback construction for hybrid control systems. *Journal of Optimization Theory and Applications*, 165(2):507–531, 2014.

[13] K. Ishii. Viscosity solutions of nonlinear second order elliptic PDEs associated with impulse control problems ii. *Funkcialaj Ekvacioj*, 38(2):297–328, 1995.

[14] E. R. Jakobsen and K. H. Karlsen. Continuous dependence estimates for viscosity solutions of fully nonlinear degenerate elliptic equations. *Electronic Journal of Differential Equations*, 2002(39):1–10, 2002.

[15] N. V. Krylov. On the rate of convergence of finite difference approximation for Bellman's equation. *St. Petersburg Mathematical Journal*, 9(3):639–650, 1998.

[16] N. V. Krylov. On the rate of convergence of finite difference approximation for Bellman's equation with variable coefficients. *Probability Theory and Related Fields*, 117(1):1–16, 2000.

[17] H. Kushner and P.G. Dupuis. *Numerical methods for stochastic control problems in continuous time*, volume 24 of *Stochastic Modelling and Applied Probability*. Springer, 2001.

[18] J. A. Sethian and A. Vladimirsky. Ordered upwind methods for hybrid control. In C. J. Tomlin and M. R. Greenstreet, editors, *Hybrid Systems: Computation and Control*, volume 2289. Springer, 2002.